mock exam math 100

29 November 2019

question 1 [10 points] Let $f(x) = x^4 - 2x^3 + 1$. To answer the following questions, you might find it helpful to first draw a sketch of the function $f(x)$.

solution:

$$f'(x) = 4x^3 - 6x^2 = 2x^2 \cdot (2x - 3)$$
$$f''(x) = 12x^2 - 12x = 12x \cdot (x - 1)$$

x	$x <$	0	$< x <$	$\frac{3}{2}$	$< x$
$f'(x)$	$-$	0	$-$	0	$+$

x	$x <$	0	$< x <$	1	$< x$
$f''(x)$	$+$	0	$-$	0	$+$

(a) Find the x-coordinates of **all** ...

- ...critical points: $0, \frac{3}{2}$ [1 point]
- ...local minima: $\frac{3}{2}$ [1 point]
- ...local maxima: none [1 point]
- ...inflection points: $0, 1$ [1 point]

If no such points exist, answer "none". You do not need to justify your answers for this part.

(b) How many real roots does $f(x)$ have? Justify your answer! [3 points]

solution: First note that

$$f(\tfrac{3}{2}) = \frac{81}{16} - 2 \cdot \frac{27}{8} + 1 = \frac{81-108+16}{16} = -\frac{11}{16} < 0.$$

Now, $f(x)$ is decreasing for $x < \frac{3}{2}$ and increasing for $x > \frac{3}{2}$, so there is *at most* one root on either side of $\frac{3}{2}$. Also, $f(x)$ is continuous and

$$f(0) = 1 > 0 \quad \text{and} \quad f(2) = 1 > 0,$$

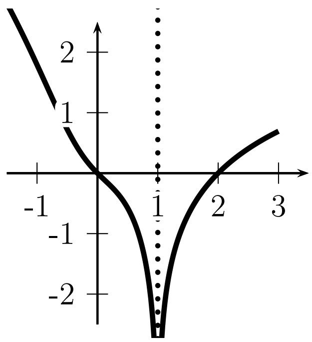
so by the IVT, there exists *at least* one root on either side. In summary, there exists *exactly* one root on either side of $\frac{3}{2}$, ie two roots in total.

Alternatively we can also argue with

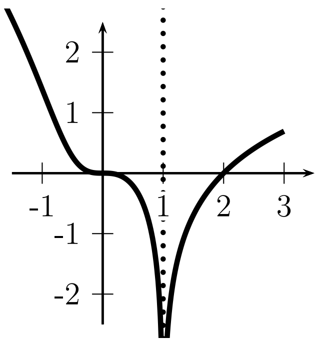
$$\lim_{x \to \pm\infty} f(x) = +\infty.$$

Furthermore, note that the root to the left of $\frac{3}{2}$ is obviously $x = 1$.

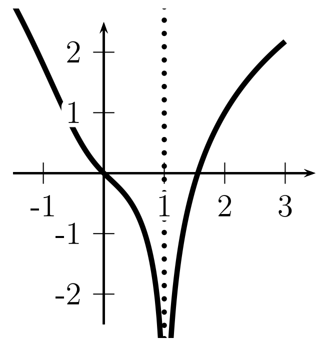
(c) Which of the following graphs shows $\log(f(x))$? You do not need to justify your answer for this part of the question. [3 points]



(a)

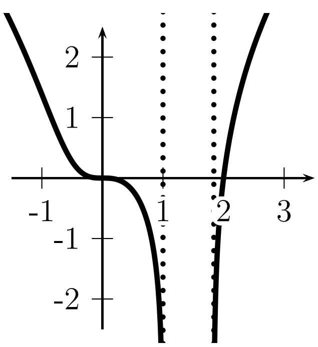


(b)

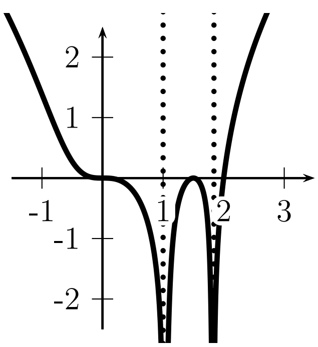


(c)

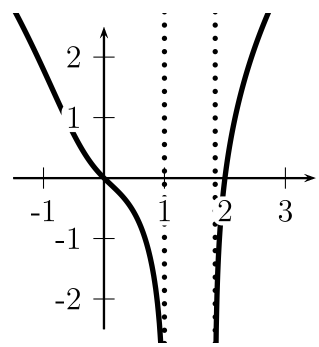
(d)



(e)



(f)



solution: By part (b), $f(x)$ has two roots, so $\log f(x)$ has two vertical asymptotes. This eliminates graphs (a), (b), and (c). Also, $f(x)$ is negative between the two roots, which eliminates (e). The only difference between (d) and (f) is the slope at $x = 0$. But

$$\frac{\mathrm{d}}{\mathrm{d}x} \log f(x) = \frac{f'(x)}{f(x)},$$

So $\log f(x)$ has a critical point at $x = 0$ like $f(x)$, which eliminates graph (f). So answer (d) is correct.

question 2 [5 points] Find all solutions of $\sqrt{3}\cos(x) - \sin(x) = 1$. (*Hint: What is a solution of* $\cos(x) = \frac{\sqrt{3}}{2}$*?*)

solution: Write
$$\tfrac{\sqrt{3}}{2}\cos(x) - \tfrac{1}{2}\sin(x) = \tfrac{1}{2}.$$
Now note that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and $\sin(\frac{\pi}{6}) = \frac{1}{2}$, so
$$\cos(\tfrac{\pi}{6})\cos(x) - \sin(\tfrac{\pi}{6})\sin(x) = \tfrac{1}{2}.$$
The LHS is $\cos(x + \frac{\pi}{6})$, so we want to find all solutions of $\cos(x + \frac{\pi}{6}) = \frac{1}{2}$. These are
$$x + \tfrac{\pi}{6} = \pm \underbrace{\arccos(\tfrac{1}{2})}_{\frac{\pi}{3}} + 2\pi n \quad \text{for any integer } n$$
ie
$$x = -\tfrac{\pi}{6} \pm \tfrac{\pi}{3} + 2\pi n \quad \text{for any integer } n$$

question 3 [5 points] Find the antiderivative of the function $y = x \cdot e^{1-x^2}$ which passes through the origin $(0, 0)$.

solution: Let's try the following candidate:
$$f(x) = e^{1-x^2}.$$
Then
$$f'(x) = -2x \cdot e^{1-x^2}.$$
Ok, this did not quite work out, but we are only off by a factor of (-2). So
$$g(x) = -\tfrac{1}{2}e^{1-x^2}$$
is an antiderivative! We find *all* antiderivates by adding a constant C:
$$g(x) = -\tfrac{1}{2}e^{1-x^2} + C.$$
Which one are we looking for? Well, we need $g(0) = 0$, so
$$0 = g(0) = -\tfrac{1}{2}e^{1-0^2} + C,$$
ie $C = \frac{1}{2}e$. Answer: $g(x) = -\frac{1}{2}e^{1-x^2} + \frac{1}{2}e = \frac{1}{2}e(1 - e^{-x^2})$.

question 4 [5 points] If

$$f(x) = \arctan(x) - \arctan\left(\frac{x-1}{x+1}\right),$$

find $f'(x)$. Hence, or otherwise, find a simple expression for $f(x)$.

solution: Simplifying $f'(x)$ gives $f'(x) = 0$. The function is defined everywhere except $x = -1$. So $f(x)$ is a constant function for $x > -1$ and $x < -1$, *but these could be different constants!* Indeed,

$$\lim_{x \to -1^-} f(x) = \arctan(-1) - \lim_{y \to +\infty} \arctan(y) = -\tfrac{\pi}{4} - \tfrac{\pi}{2} = -\tfrac{3\pi}{4}$$

and similarly

$$\lim_{x \to -1^+} f(x) = \arctan(-1) - \lim_{y \to -\infty} \arctan(y) = -\tfrac{\pi}{4} + \tfrac{\pi}{2} = \tfrac{\pi}{4}$$

So

$$f(x) = \begin{cases} \frac{\pi}{4} & \text{for } x > -1 \\ -\frac{3\pi}{4} & \text{for } x < -1 \end{cases}$$