Antiderivatives

so far: Given a function $f(x)$, find $\frac{df}{dx}(x)$.

opposite problem: Given a function $f(x)$, find a function $F(x)$ such that

$$\frac{dF}{dx}(x) = f(x).$$

In that case, $F(x)$ is called an antiderivative of $f(x)$.

examples:

- $f(x) = 3x^2$ $\boxed{F(x) = x^3}$
- $f(x) = 4x^4 + 2$ $\boxed{F(x) = \frac{4}{5}x^5 + 2x}$
- $f(x) = \frac{1}{x^2}$ $\boxed{F(x) = -\frac{1}{x}}$
- $f(x) = \frac{1}{x}$ $\boxed{F(x) = \log|x|}$
- $f(x) = \sin(x)$ $\boxed{F(x) = -\cos(x)}$
- $f(x) = \cos(4x)$ $\boxed{F(x) = \frac{1}{4}\sin(4x)}$

<u>observation:</u> If $F(x)$ is an antiderivative of $f(x)$, then $(F(x)+c)$ is also an antiderivative of $f(x)$.

<u>In fact:</u> Suppose $F(x)$ and $G(x)$ are antiderivatives of $f(x)$. Then, (by definition)

$$\frac{d}{dx}F(x) = f(x) \quad \text{and} \quad \frac{d}{dx}G(x) = f(x).$$

Hence

$$\frac{d}{dx}\Big(G(x) - F(x)\Big) = \frac{d}{dx}G(x) - \frac{d}{dx}F(x) = f(x) - f(x) = 0.$$

Now, by a corollary of the MVT, it follows that

$$G(x) - F(x) = c \quad \text{for some } c \in \mathbb{R}.$$

In other words:

<u>Theorem:</u> If $F(x)$ is an antiderivative of $f(x)$, then <u>all</u> other antiderivatives are given by

$$F(x) + c\,, \quad c \in \mathbb{R}.$$

So up to this ambiguity, we may talk about <u>the</u> antiderivative of a function.

Finding antiderivatives is <u>hard</u>, generally speaking.

$\boxed{\text{(For some techniques, see next term.)}}$

<u>Our method of choice:</u> Read differentiation tables upside down

$F(x)$	x^n	$\sin x$	$\tan x$	$\cos x$	e^x	$\log\lvert x\rvert$	$\arcsin x$	$\arctan x$
$f(x) = F'(x)$	$n \cdot x^{n-1}$	$\cos x$	$\frac{1}{\cos^2(x)}$	$-\sin x$	e^x	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

<u>examples:</u>

- $f(x) = x^m$ <u>first guess:</u> $F(x) = x^{m+1} + c$

  $\hookrightarrow$ <u>test:</u> $\frac{d}{dx} F(x) = (m+1) \cdot x^m$

  close, but not quite what we wanted!

  <u>second guess:</u> $F(x) = \frac{1}{m+1} \cdot x^{m+1} + c$

  $\hookrightarrow$ <u>test:</u> $\frac{d}{dx} F(x) = \cancel{\frac{1}{m+1}} \cdot \cancel{(m+1)} \cdot x^m$ ✓

- $f(x) = \frac{1}{x^2}$ ~~first~~ second <u>guess:</u> $F(x) = -\frac{1}{x} + c$

  $\hookrightarrow$ <u>test:</u> $\frac{d}{dx} F(x) = +\frac{1}{x}$ ✓

- $f(x) = \dfrac{1}{1+9x^2}$ first guess: $F(x) = \arctan(3x) + c$

  ↳ test: $\dfrac{d}{dx} F(x) = \dfrac{1}{1+(3x)^2} \cdot 3$

  second guess: $F(x) = \dfrac{1}{3} \cdot \arctan(3x) + c$ ✓

- $f(x) = \dfrac{2x}{1+x^2}$?

  observation: $2x = \dfrac{d}{dx}(1+x^2)$

  also recall: $\dfrac{d}{dx}\left(\log|g(x)|\right) = \dfrac{g'(x)}{g(x)}$

  so a good guess might be $F(x) = \log(\underbrace{1+x^2}_{>0}) + c$. ✓

  = informed

- $f(x) = \dfrac{4x^4 + x^2 - 8x + 1}{1+4x^2}$?

  idea: Simplify!

  $$f(x) = \frac{x^2(4x^2+1) - 8x + 1}{1+4x^2} = x^2 - \frac{8x}{1+4x^2} + \frac{1}{1+4x^2}$$

  Now find the antiderivatives of those three pieces separately!

  $$F(x) = \frac{1}{3}x^3 - \log(\underbrace{1+4x^2}_{>0}) + \frac{1}{2}\arctan(2x) + c$$

<u>example:</u> Recall the differential equation

$$\frac{d}{dt} Q(t) = -k \cdot Q(t) \qquad (*)$$

for radioactive decay, where $Q(t)$ is an amount of radioactive material at time t (so $Q(t) > 0$).

Since $Q(t) \neq 0$, we may rewrite $(*)$ as

$$\frac{\frac{d}{dt} Q(t)}{Q(t)} = -k$$

Notice that the LHS is equal to $\frac{d}{dt} \log(\underbrace{Q(t)}_{>0})$; so in other words, the antiderivative of the LHS is $\log(Q(t))$. The antiderivative of $-k$ is $-k \cdot t$, so $-k \cdot t$ and $\log(Q(t))$ have to agree up to a constant:

$$\log(Q(t)) = -k \cdot t + c \qquad \text{for some } c \in \mathbb{R}$$

Hence

$$Q(t) = e^{-k \cdot t + c} = \underbrace{e^{c}}_{Q(0)} \cdot e^{-k \cdot t} = Q(0) \cdot e^{-k \cdot t}.$$

This is precisely the solution to $(*)$ from earlier this term!

$\boxed{\text{teaching evaluations!}}$