Immersed curves in Khovanov homology

by Claudius Zibrowius

Lecture 1: Bar-Natan's tangle invariant†.

- Definition: Cob (no gradings)
- Definition: $[\![T]\!]$ via cube of resolutions $\leftrightarrow$ Example: 2-crossing tangle
- Definition: complexes over a category
- Example: Reidemeister I move
- Definition: $\mathrm{Cob}_{/l}$ (no gradings)
- Theorem: invariance of $[\![T]\!]_{/l}$ (sketch Reidemeister I move)
- Lemma: delooping and its consequences for $[\![T]\!]_{/l}$ of 2- and 4-ended tangles

Lecture 2: Bases of morphism spaces‡.

- Definition: the variable H and dotted cobordisms
- Theorem: simple cobordisms form a free basis over $\mathbb{Z}[H]$
- Example: algebra for 2-ended and 4-ended tangles
- Theorem: quiver description for algebra for 4-ended tangles
- Definition: $Д(T)$ $\leftrightarrow$ Example: n-twist rational tangle
- Theorem: reduced Bar-Natan and Khovanov homology as representable functors

Lecture 3: Immersed curve invariants‡.

- Definition: geometric interpretation of $\mathcal{B}$
- Definition: geometric interpretation of complexes over $\mathcal{B}$
- Theorem: classification for objects by examples
- Definition: local systems by examples
- Theorem: the immersed curves $\widetilde{\mathrm{BN}}(T)$
- Example: n-twist tangles $\leftrightarrow$ Theorem: $D^2 \smallsetminus (3\text{ points}) \mathrel{\hat{=}} \partial B^3 \smallsetminus \partial T$
- Theorem: Conway mutation and the dependence of $\widetilde{\mathrm{BN}}(T)$ on the basepoint $*$

Lecture 4: Pairing and applications‡.

- Theorem: algebraic pairing for $\widetilde{\mathrm{BN}}$ and $\widetilde{\mathrm{Kh}}$
- Theorem: classification for morphism spaces
- Theorem: geometric pairing for $\widetilde{\mathrm{BN}}$ and $\widetilde{\mathrm{Kh}}$
- Theorem: mutation invariance of $\widetilde{\mathrm{BN}}$ and $\widetilde{\mathrm{Kh}}$ over characteristic 2
- Example: mutation invariance with signs is false
- Theorem: mutation invariance of Rasmussen's s-invariant over any field

References.

Main references:

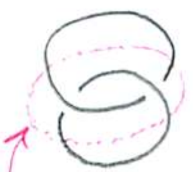
†D. Bar-Natan, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. (9), 2005, 1443–1499. (arXiv:math/0410495)

‡A. Kotelskiy, L. Watson and C. Zibrowius, *Immersed curves in Khovanov homology*, In preparation.

Lecture 1: Bar-Natan's tangle invariant

<u>main goal:</u> geometric interpretation of a "local" version of Khovanov homology

<u>Khovanov:</u> $\{\text{links}\} \longrightarrow \dfrac{\{\text{chain complexes}/\mathbb{Z}\}}{\text{homotopy}}$



$L \longmapsto CKh(L)$

<u>question:</u> What is the Khovanov theory of this kind of object?

<u>Bar-Natan:</u> $\{\text{tangles}\} \longrightarrow \dfrac{\{\text{chain complexes}/\mathrm{Cob}_{/\ell}\}}{\text{homotopy}}$

$T \longmapsto [\![T]\!]_{/\ell}$

<u>plan for today:</u>
- $\mathrm{Cob}_{/\ell}$
- chain complexes over a category
- construction of $[\![T]\!]_{/\ell}$

<u>remarks:</u>
- website / lecture notes / references
- → exercises + cheat sheet ←

def: $B := \{2m \text{ points}\} \subset \partial D^2$, $m > 0$, eg $B =$  .

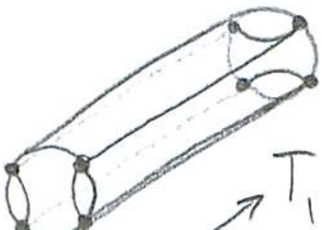
category $Cob := Cob(B)$:

ob: crossingless tangles with ends $= B$ / isotopy fixing ∂D^2

eg  ,  , 

mor: formal linear combination of $\underbrace{\text{cobordisms } T_0 \longrightarrow T_1}$

 $= T_0 \longrightarrow T_1 =$ 



T_1

T_0

$=$ orientable surface Σ together with an identification of $\partial\Sigma$ with

$$\underbrace{(T_0 \times \{0\}) \cup (B \times [0,1]) \cup (T_1 \times \{1\})}_{\cap}$$

$$\partial(D^2 \times [0,1]) = (D^2 \times \{0\}) \cup (\partial D^2 \times [0,1]) \cup (D^2 \times \{1\})$$

composition: concatenation

$$T_0 \xrightarrow{g} T_1 \xrightarrow{f} T_2 = T_0 \xrightarrow{f \circ g} T_2$$

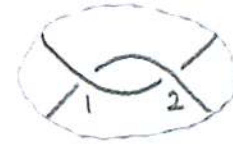
identity: $id: T \xrightarrow{T \times [0,1]} T \quad \forall T \in ob\, Cob.$

notation: This cobordism with $\Sigma = D^2$ is called a saddle cobordism. We denote it by

 :  $\longrightarrow$  .

construction of $[\![T]\!]$:

1) Number all crossings of T

$T =$ 

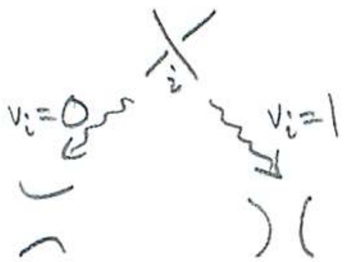
2) Let $n = \#\{\text{crossings of } T\}$.

$n = 2$

For $v \in \{0,1\}^n$, define

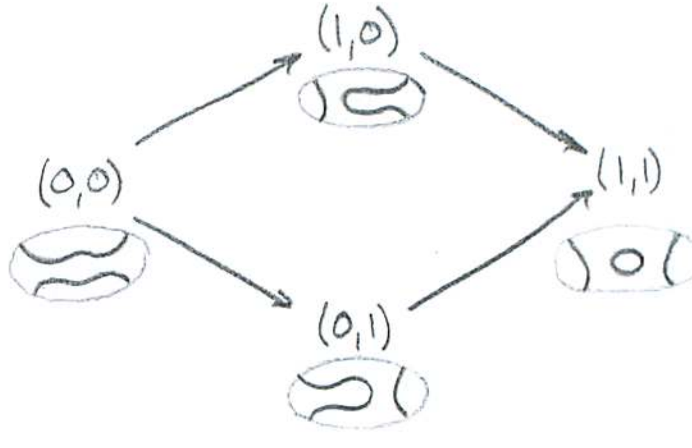
$v = (0,1)$

$v(T) :=$ crossingless tangle:



$v(T) =$ 

3) Label the vertices $v \in \{0,1\}^n$ of the n-dimensional cube $[0,1]^n$ by $v(T)$.

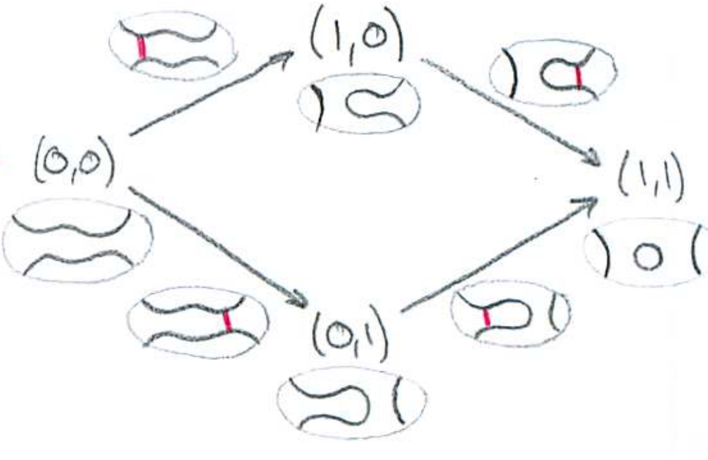


4) Label each edge

$v = (\_,0,\_) \longrightarrow (\_,1,\_) = w$

by the saddle cobordism.

The result is $[\![T]\!]$.



remark: We work over $\mathbb{F} = \mathbb{Z}/_2$; in general, we need to add signs to edges.

<u>def:</u> A (chain) complex over a category $\mathcal{C}$ is a pair (C, d), where

$$C = \bigoplus_{i \in I} X_i \quad , \quad I = \text{finite index set}, \ X_i \in \text{ob}\, \mathcal{C}$$

and $d : C \longrightarrow C$ is a matrix $(d_{ji})_{ji}$ of morphisms

$$d_{ji} : X_i \longrightarrow X_j \quad \text{in } \mathcal{C}$$

such that $d^2 = 0$.

<u>example:</u> For a field k, let $\mathcal{C} = \bullet \circlearrowright k = \left(\underline{\text{ob}} : \{\bullet\}, \text{mor} : \text{Mor}(\bullet, \bullet) = k\right)$

Then $\{\text{chain complexes}/\mathcal{C}\} \cong \{\text{chain complexes of vector spaces}/k\}$ (+ basis)

> <u>remark:</u> This is actually an equivalence of categories.
> → see Q1 on exercise sheet ←

<u>subexample:</u> cellular homology

CW complex

$X :=$ 

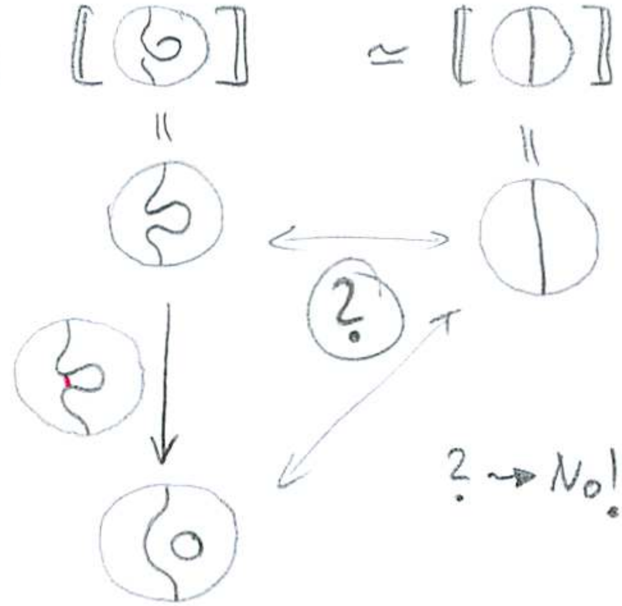
+ attach a 2-cell a along $e_1 e_2^2$

$$\rightsquigarrow C^{\text{cell}}(X) = \begin{tikzcd} & e_1 \arrow[dr, "\cancel{0}"] & \\ a \arrow[ur, "1"] \arrow[dr, "2"'] & & v \\ & e_2 & \end{tikzcd}$$

$\boxed{\text{compare with } [\![T]\!]}$

> <u>remark:</u> One should also add gradings to the definition above.

<u>example:</u>



? ⇝ No! (consider )

<u>def:</u> $\mathrm{Cob}_{/\ell} :=$ Cob modulo the following relations on morphisms:

(S)  $= 0$

(4Tu) 

→ Bar-Natan also included the T-relation  $= 2$, but it follows from the other two if $2 \neq 0$. (Q 4) ←

<u>def/thm:</u> [Bar-Natan '04]

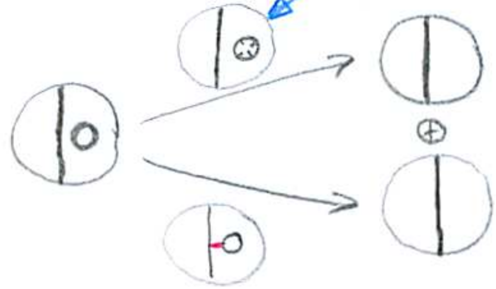
$[\![T]\!]_{/\ell} := [\![T]\!]$ considered as a complex over $\mathrm{Cob}_{/\ell}$.

Then, up to chain homotopy, $[\![T]\!]_{/\ell}$ is a tangle invariant.

⊗ = cap off circle with a disc

lemma: (delooping)

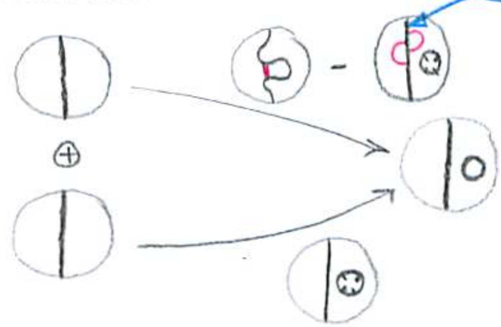
Over $Cob_{/\ell}$, $\dot{\imath}$:



is an isomorphism with inverse

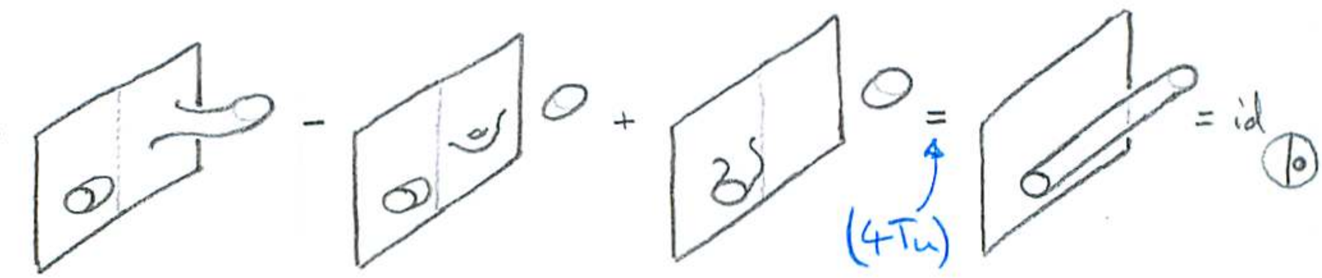
genus 1 surface

$\dot{\imath}^{-1}$:



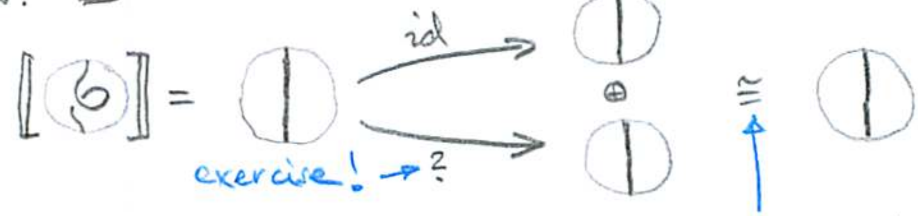
proof:

$\dot{\imath}^{-1} \circ \dot{\imath} =$



$\dot{\imath} \circ \dot{\imath}^{-1}$ is similar. ∎

consequence 1:



→ cancellation (Q 2,3) ←

consequence 2:

Delooping allows us to write any complex over $Cob_{/\ell}$ in terms of crossingless tangles <u>without closed components</u> and morphisms between them.

examples: $B =$  ⇒ single object 

$B =$  ⇒ two objects  and 

Lecture 2: Bases of morphism spaces

Last time: delooping $\Rightarrow$ it suffices to understand the full subcategory of $\mathrm{Cob}_{/\ell}$ generated by crossingless tangles without closed components.

notation: [Kotelskiy–Watson–Z]

Mark one of the points in B by $*$, eg  or .

This picks out a special component of each cobordism in $\mathrm{Cob}_{/\ell}$.

Let us write

$H \cdot$ 

and 

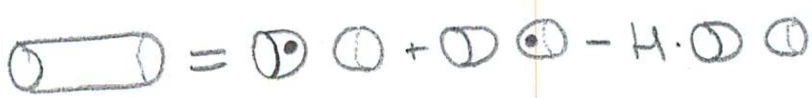
prop: [KWZ]

In $\mathrm{Cob}_{/\ell}$, the following relations hold:

($S_\bullet$)  $= 1$

(0-trading)  $= 0$

(H-trading) 

(neck-cutting) 

◀ proof: exercise ▶

$\boxed{\text{Remark: comparison with Bar-Natan's } Cob_\bullet}$

def: A dotted cobordism is simple if it does not contain a closed component and every open component is a disc containing at most one dot and in the case of the special component no dots.

thm: [reformulation of a result of Naot '06]
For any two objects T and T' of $Cob_{/\ell}$,

$$Mor_{Cob_{/\ell}}(T,T') = \mathbb{Z}[H]\langle \text{simple cobordisms} \rangle.$$

Note: There are $2^{\#\{\text{components of } T \cup T'\}-1}$ simple cobordisms.

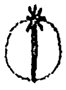
proof:
- Use neck-cutting to minimize the genus.
- Use O- and H-trading to minimize $\#\{\text{dots}\}$.
- Use S- and $S_\bullet$-relation to remove closed components.

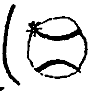
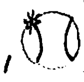
free generation: more involved

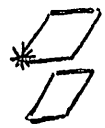
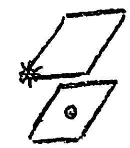
↳ Naot uses pairing thm + alg. def. of Kh/BN

↳ There should also be a purely combinatorial argument.

examples:

$\mathrm{Mor}_{Cob_{/l}}\Big($  ,  $\Big) = \mathbb{Z}[H]\langle \mathrm{id}$  $\rangle$

$\mathrm{Mor}_{Cob_{/l}}\Big($  ,  $\Big) = \mathbb{Z}[H]\langle$  $\rangle$

$\mathrm{Mor}_{Cob_{/l}}\Big($  ,  $\Big) = \mathbb{Z}[H]\langle \mathrm{id} =$  ,  $=$  $\rangle$

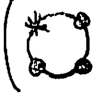
def: $\mathcal{B}$ = quiver algebra of

$$\begin{tikzcd} \bullet \arrow[loop left, "D"] \arrow[r, bend right, "S"'] & \circ \arrow[l, bend right, "S"'] \arrow[loop right, "D"] \end{tikzcd} \Big/ DS = 0 = SD$$

ie. $\mathcal{B} = \mathbb{Z}\langle$ paths in this quiver $\rangle$ as an Abelian group,
composition = concatenation (if possible, otherwise 0)

so eg: $\mathrm{Mor}_{\mathcal{B}}(\bullet, \circ) = \mathbb{Z}\langle S^{2n+1} \mid n \geq 0\rangle$

$\mathrm{Mor}_{\mathcal{B}}(\bullet, \bullet) = \mathbb{Z}\langle S^{2n}, \mathrm{id}_\bullet, D^n \mid n > 0\rangle$

thm: [KWZ]
The functor $\mathcal{B} \longrightarrow Cob_{/l}\big($  $\big)$

$\bullet \longmapsto$  $\qquad \circ \longmapsto$ 

$S \longmapsto$ saddle cob. $\quad D \longmapsto$ dot cob.

is well-defined and induces an isomorphism between $\mathcal{B}$ and the full subcategory of $Cob_{/l}$ generated by  and .

<u>proof:</u>

<u>well-defined:</u> the saddle and dot cobordisms compose to 0, since

$$\boxed{*\bullet} = 0 .$$

<u>fully faithful:</u> Note that

$D \longmapsto$  $\longleftarrow SS + H \cdot id$,

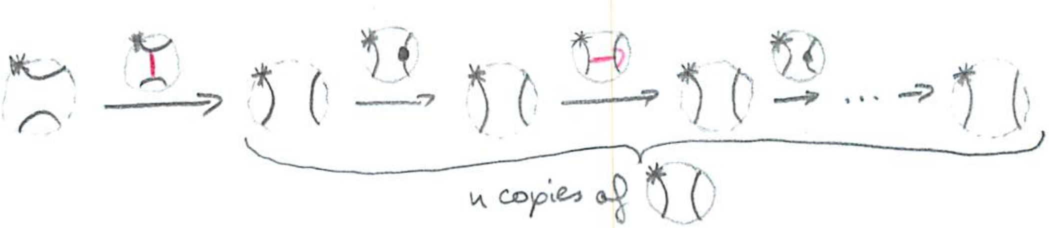
so we define $H := D - SS \in \mathcal{B}$.

Now observe that the morphism spaces are freely generated over $\mathbb{Z}[H]$ by the identity, D/dot cob. and S/saddle cob. ■

→ <u>exercise (Q7)</u>: This isomorphism depends on the basepoint $*$. How? ←

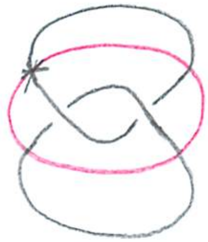
<u>def:</u> Given a pointed (ie with a distinguished tangle end) 4-ended tangle T, let $\Delta(T)$ be the complex over $\mathcal{B}$ corresponding to $[\![T]\!]_{/2}$ under the isomorphism above.

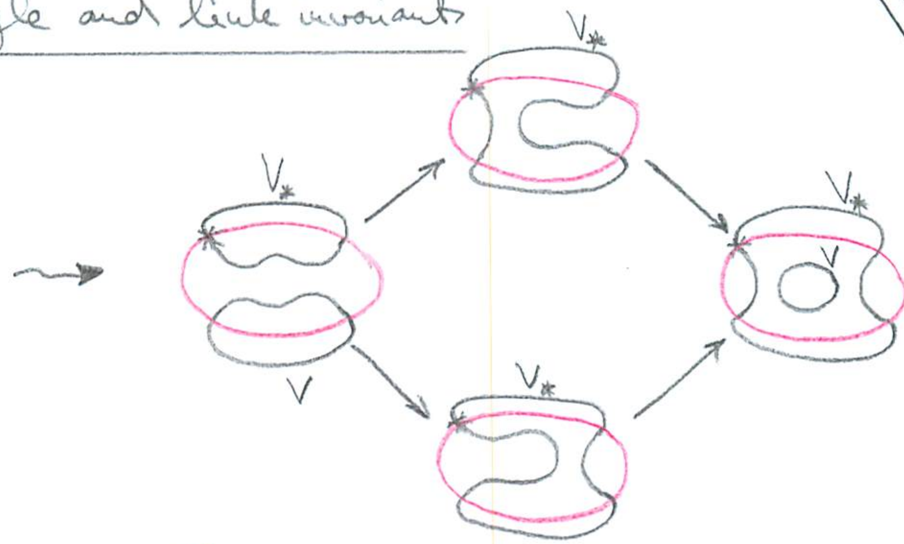
<u>example:</u> $T_n =$  , $n \geq 1$ → see exercise (Q6) ←

$[\![T_n]\!]_{/2} =$ 

$$\Delta(T_n) = \bullet \xrightarrow{S} \circ \xrightarrow{D} \circ \xrightarrow{SS} \circ \xrightarrow{D} \cdots \longrightarrow \circ$$

Relationship between tangle and link invariants

$L =$ 



1) cube of resolutions
2) Label the special component $\circledast$ by V_* and all other components by V (see cheat sheet).
3) Take $\otimes_{\mathbb{Z}[H]}$ at each vertex.
4) Add edge maps: Δ, m, Δ_*, m_* (see cheat sheet).

TQFT

$$\begin{tikzcd}
 & V_* \otimes V^{\otimes 0} \arrow[dr] & \\
V_* \otimes V^{\otimes 1} \arrow[ur] \arrow[dr] & & V_* \otimes V^{\otimes 1} \\
 & V_* \otimes V^{\otimes 0} \arrow[ur] &
\end{tikzcd}$$

The result is $\widehat{CBN}(L)$, the reduced Bar-Natan chain complex. Also, the reduced Khovanov chain complex $\widehat{CKh}(L)$ is equal to $\widehat{CBN}(L)/(H=0)$.

<u>observation</u>: Inside , the cube of resolution agrees with $[\![T]\!]$!

<u>question:</u> Can I replace the TQFT by a functor

$$\mathrm{Cob}_{/\ell} \longrightarrow \mathrm{Mod}^{\mathbb{Z}[H]}$$ ← category of $\mathbb{Z}[H]$-modules

such that $[\![T]\!]_{/\ell} \longmapsto \widetilde{\mathrm{CBN}}(T)$.

<u>candidate:</u>

$$\mathrm{Mor}(\;,-): \quad \mathrm{ob}\,\mathrm{Cob}_{/\ell} \ni T \longmapsto \mathrm{Mor}(\;,T)$$

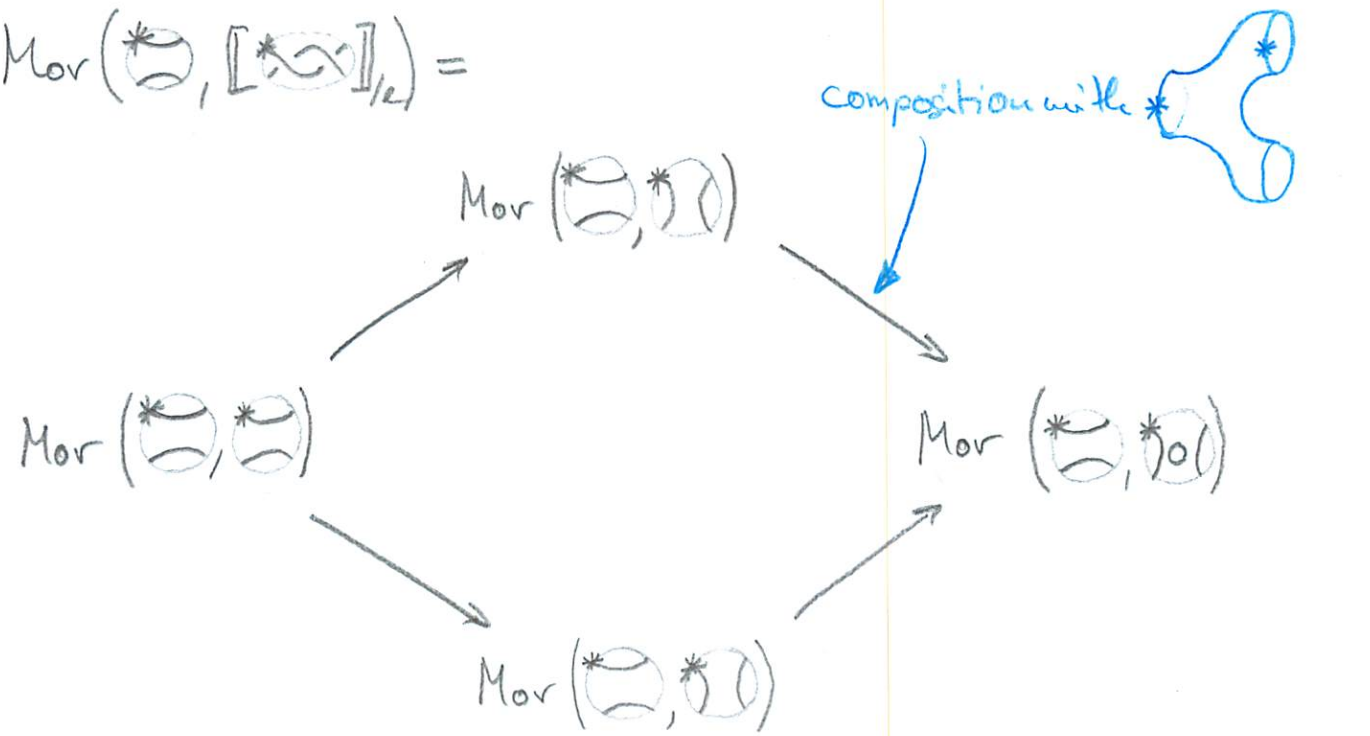
$$(T \xrightarrow{f} T') \longmapsto \left(\mathrm{Mor}(\;,T) \longrightarrow \mathrm{Mor}(\;,T')\right)$$

$$g \longmapsto (f \circ g)$$

<u>thm:</u> $\widetilde{\mathrm{CBN}}(T) = \mathrm{Mor}(\;,[\![T]\!]_{/\ell})$.

<u>example:</u>

$\mathrm{Mor}(\;,[\![\;]\!]_{/\ell}) =$

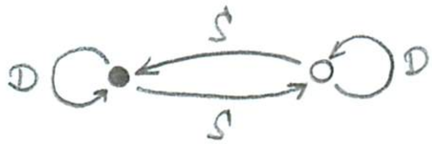


Lecture 3: Immersed curve invariants

Last time: For pointed 4-ended tangles T, $[\![T]\!]_{/\!\!/}$ is equivalent to a complex $\Delta(T)$ over $\mathcal{B}$.

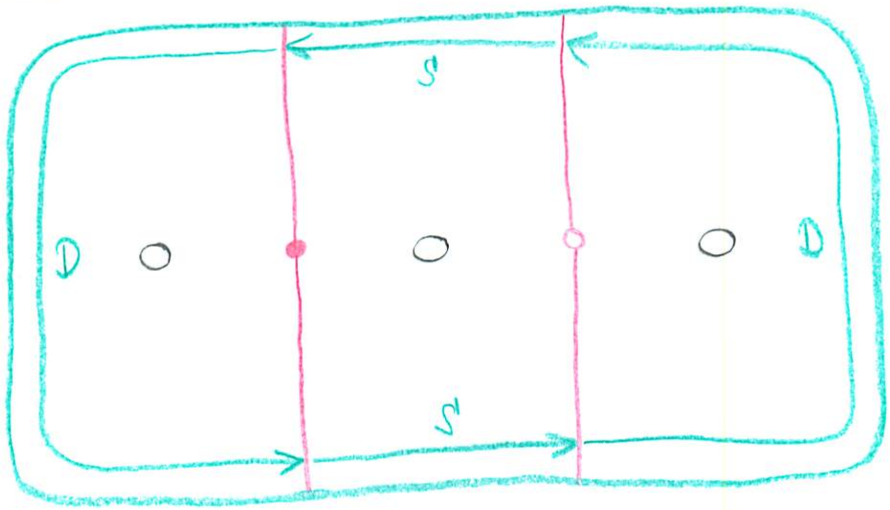
> today: classification of complexes over $\mathcal{B}$ in terms of immersed curves on $D^2 \smallsetminus (3\text{ points})$

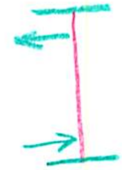
recall: $\mathcal{B}$ = path algebra of

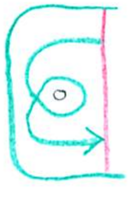
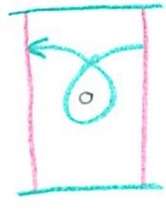


$/ \; D \cdot S = 0 = S \cdot D$

geometric interpretation:



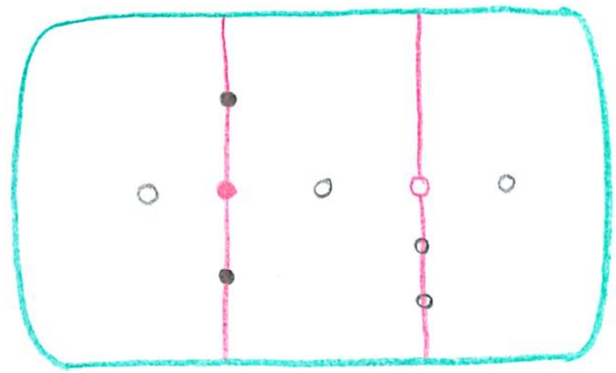
relations:  $= 0$, but  $\neq 0$

eg $D^2 =$  and $S^3 =$  .

How to represent complexes over $\mathcal{B}$ graphically:

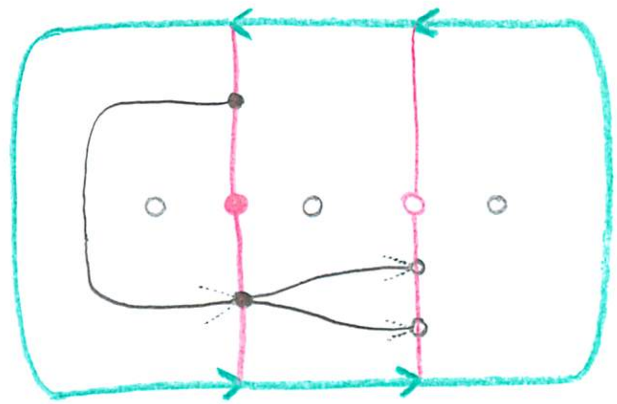
$$(C,d) = \begin{tikzcd} & & \circ \\ \bullet \arrow[r, "D"] & \bullet \arrow[ur, "S"] \arrow[dr, "S"'] & \\ & & \circ \end{tikzcd}$$

1) For each object $\bullet$ in (C,d), mark a point on . Same for $\circ$/ .

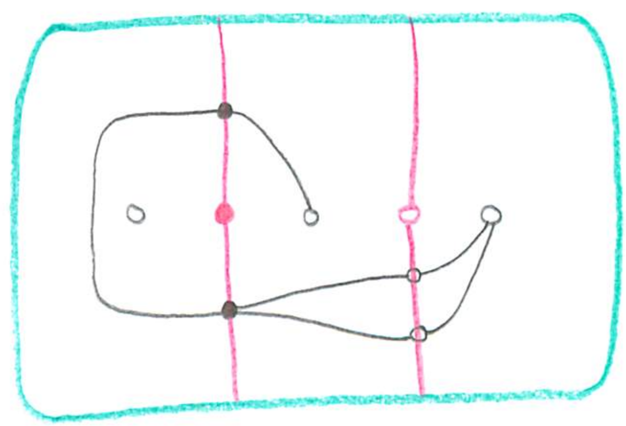


2) For each arrow labelled by a power of D or S, connect its endpoints on the arcs / by a corresponding path.

Note: These paths inherit an orientation from the boundary of the surface!

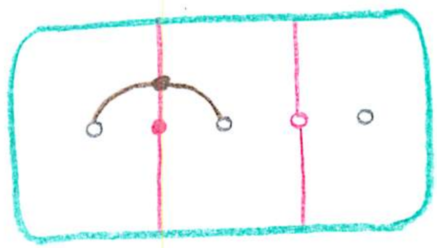


3) Connect each puncture to all adjacent generators that are not ends of paths on the same face.

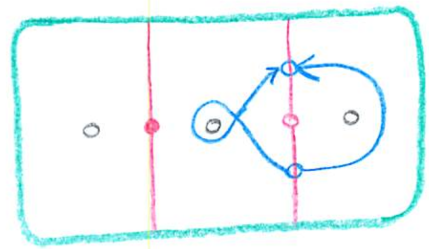


examples:

1) $\bullet$ $(d=0)$



2)
$$\begin{tikzcd} \circ \arrow[r, bend left, "D"] \arrow[r, bend right, "S^2"'] & \circ \end{tikzcd}$$



3) $\bullet \xrightarrow{1} \bullet$

?!

$\Rightarrow$ Need to start with a fully cancelled complex!

classification thm 1: [KWZ; based on HKK, HRW, Z]

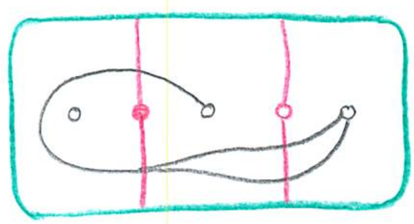
Let k be a field. Then, $\exists$ 1:1-correspondence

$$\frac{\{\text{complexes over } \mathcal{B} \text{ and } k\}}{\text{homotopy}} \overset{\varphi}{\longleftrightarrow} \frac{\{\text{immersed curves on } D^2 \setminus (3 \text{ points})\}}{\text{homotopy}}.$$

- arcs connecting punctures (like 1)
- circles (like 2) ← can carry local systems

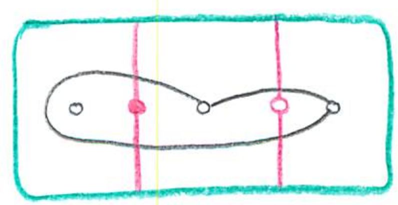
example:

$$\begin{tikzcd} & & \circ \\ \bullet \arrow[r, "D"] & \bullet \arrow[ur, "S"] \arrow[dr, "S"'] & \\ & & \circ \end{tikzcd}$$



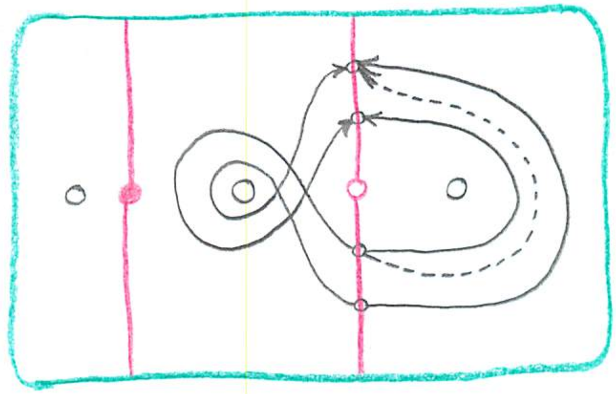
Clean-Up Lemma (Q8) $\longrightarrow \cong$

$$\bullet \xrightarrow{D} \bullet \xrightarrow{S} \circ$$
$$\oplus$$
$$\circ$$



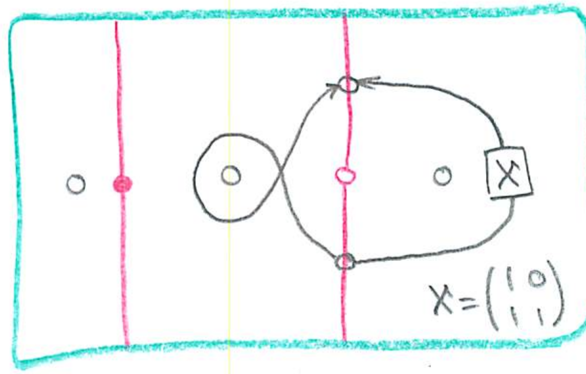
def by example: (local systems)

$$\begin{tikzcd} \circ \arrow[r, bend left, "S^2"] \arrow[r, bend right, "D"'] \arrow[dr, dashed, "D"] & \circ \\ \circ \arrow[r, bend left, "S^2"] \arrow[r, bend right, "D"'] & \circ \end{tikzcd}$$



$$\|$$

$$\begin{array}{c} \circ \\ \oplus \\ \circ \end{array} \begin{tikzcd} {} \arrow[r, bend left, "\begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}"] \arrow[r, bend right, "\begin{pmatrix} D & 0 \\ D & D \end{pmatrix}"'] & {} \end{tikzcd} \begin{array}{c} \circ \\ \oplus \\ \circ \end{array}$$

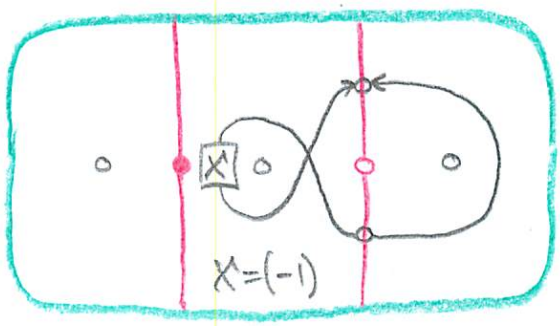


so: local system = decoration of an immersed curve by an invertible matrix $\in GL_n(k)$, considered up to matrix similarity.

Note: Local systems may also record signs:

eg $\circ \xrightarrow{H} \circ$

(recall $H = D - S^2$)



$$\boxed{\text{Note: Local systems only occur on immersed circles, not on arcs.}}$$

def: For pointed 4-ended tangles T, define

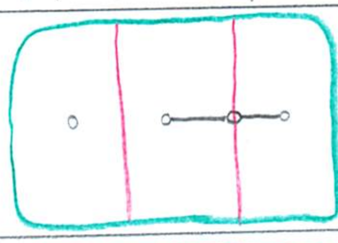
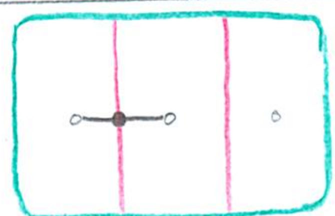
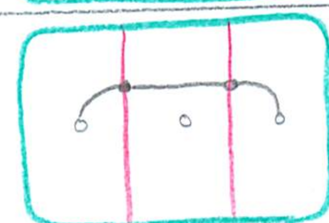
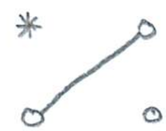
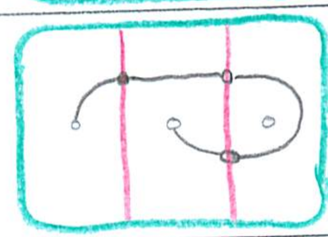
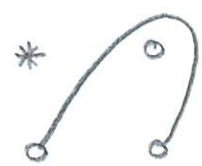
$$T \longmapsto [\![T]\!]_{/\ell} \longleftrightarrow \Delta(T) \overset{\varphi}{\longleftrightarrow} : \widetilde{\mathrm{BN}}(T) = \widetilde{\mathrm{BN}}(T;k)$$

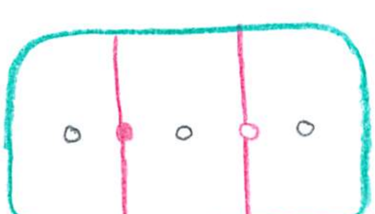
(lecture 1) (lecture 2) over a field k

"reduced Bar-Natan invariant of T"

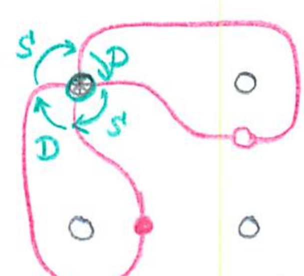
or simply "arc invariant of T"

examples:

T	$\Delta(T)$	$\widetilde{\mathrm{BN}}(T)$ on $D^2 \setminus (3\text{ points})$	$\widetilde{\mathrm{BN}}(T)$ on $(S^2 \setminus (4\text{ points}), *)$
	$\circ$		
	$\bullet$		
	$\circ \xrightarrow{S} \bullet$		
	$\circ \xrightarrow{D} \circ \xrightarrow{S} \bullet$		
	$\circ \xrightarrow{S^2} \circ \xrightarrow{D} \circ \xrightarrow{S} \bullet$	?	?



$\longleftrightarrow$



$D^2 \setminus (3\text{ points}) \longleftrightarrow (S^2 \setminus (4\text{ points}), *)$

<u>thm:</u> [KWZ]

For any $\rho \in MCG(S^2 \setminus (4 \text{ points}))$, $\widetilde{BN}(\rho T; \mathbb{F}_2) = \rho \, \widetilde{BN}(T; \mathbb{F}_2)$

<u>conj:</u> This also holds over other fields.

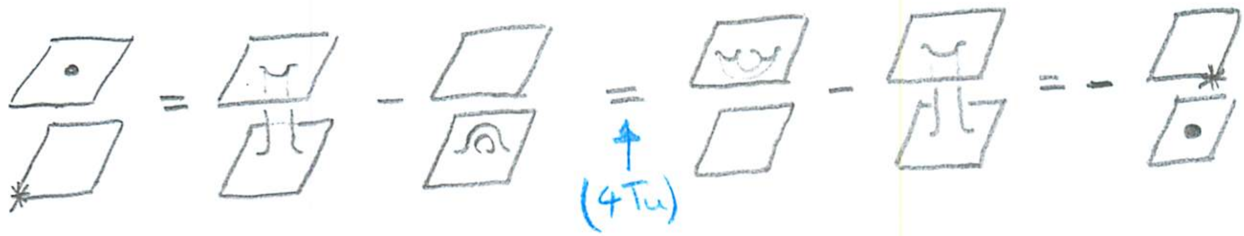
<u>observation:</u> $\widetilde{BN}(T)$ depends on the basepoint $*$.

<u>recall:</u> $*$ determines the special component of a cobordism.

→ saddle: single component, so $*$ is irrelevant.

→ identity: no dots, so $*$ is irrelevant.

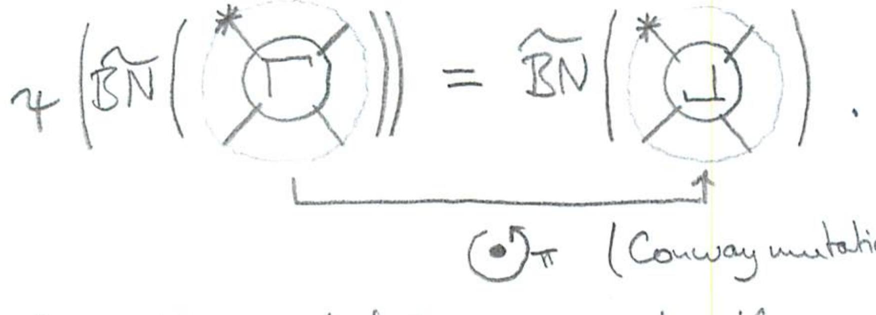
→ dot cob:



<u>Cor:</u> [KWZ]

Let $\psi: \mathcal{B} \longrightarrow \mathcal{B}$, $S \longmapsto S$, $D \longmapsto -D$.

Then for any 4-ended tangle T,



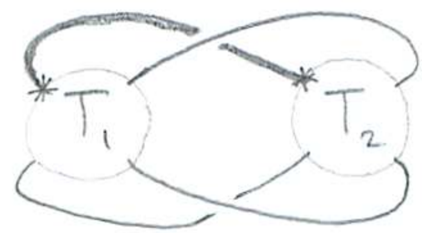
(Conway mutation)

In particular, the underlying curves stay the same, and so do the local systems <u>up to sign</u>.

Lecture 4: Pairing and applications

Algebraic Pairing Theorem: [Bar-Natan, Marion]

Given two pointed 4-ended tangles T_1 and T_2, let

$$\mathcal{L}(T_1, T_2) := $$  .

Then

$$\widehat{CBN}(\mathcal{L}(T_1,T_2)) \cong \mathrm{Mor}\big(\Delta(mT_1), \Delta(T_2)\big)$$

as chain complexes over $\mathbb{Z}[H]$. mirror of T_1 ($\times \leftrightarrow \times$)

Remark: Given two complexes (C,d) and (C',d') over $\mathcal{B}$,

$$\mathrm{Mor}\big((C,d),(C',d')\big) = \{\text{matrices of morphisms in } \mathcal{B}\}$$

no dependence on d nor d'!

carries a differential defined by

$$D(f) = d' \circ f + f \circ d \qquad \boxed{(\text{mod } 2;\ \text{in general, add signs here})}$$

examples:

1) $\widehat{CBN}(\text{unknot}) = \widehat{CBN}(\mathcal{L}(*\asymp, *)()) = \mathrm{Mor}(\Delta(*\asymp), \Delta(*)())$

$= \mathrm{Mor}(\bullet, \circ) = \langle S, S^3, S^5, \ldots \rangle = \mathbb{Z}[H]\langle S \rangle.$

2) $\widehat{CBN}(\text{unknot}) = \widehat{CBN}\left(\mathcal{L}(\,,\,)\right) = \mathrm{Mor}\left(\bullet,\ \bullet \xrightarrow{S} \circ\right)$

$$= \left\{ \begin{tikzcd} \bullet \arrow[r, "D^n + S^{2n'}"] \arrow[dr, "S^{2n''+1}"'] & \bullet \arrow[d, "S"] \\ & \circ \end{tikzcd} \;\middle|\; n>0,\ n',n''\geq 0 \right\}$$

$$= \{D^n \mid n>0\} \oplus \cancel{\{S^{2n'} \mid n'\geq 0\} \xrightarrow{\cdot S} \{S^{2n''+1} \mid n''\geq 0\}}$$

cancellation

$$= \mathbb{Z}[H]\langle D\rangle .$$

$\boxed{\text{\underline{exercise}: Compute } \widehat{CBN}(\,) \text{ from this tangle decomposition.}}$

Recall that the reduced Khovanov complex $\widehat{CKh}(L)$ of a link L is defined by the short exact sequence

$$0 \longrightarrow \widehat{CBN}(L) \xrightarrow{\cdot H} \widehat{CBN}(L) \longrightarrow \widehat{CKh}(L) \longrightarrow 0$$

<u>cor:</u> $\widehat{CKh}\left(\mathcal{L}(T_1,T_2)\right) \cong \mathrm{Mor}\Big(\Delta(mT_1),\ \underbrace{\Delta(T_2) \xrightarrow{H} \Delta(T_2)}_{=:\,\Delta^1(T_2)}\Big)$

<u>example:</u>

$$\widehat{CKh}(\text{unknot}) \cong \mathrm{Mor}\left(\Delta(\,),\ \Delta^1(\,)\right)$$

$$= \mathrm{Mor}\left(\bullet,\ \circ \xrightarrow{H} \circ\right) = \left\{ \begin{tikzcd} & \circ \arrow[d, "H"] \\ \bullet \arrow[r, "S"'] & \circ \end{tikzcd} \right\} = \mathbb{Z}$$

<u>classification thm 2:</u> [KWZ; based on HRW, Z]

Let $L = \varphi(C,d)$ and $L' = \varphi(C',d')$ for two complexes (C,d) and (C',d') over $\mathcal{B}$ and a field k. Then

$$H_*\Big(\mathrm{Mor}\big((C,d),(C',d')\big)\Big) \cong \underbrace{HF(L,L')}$$

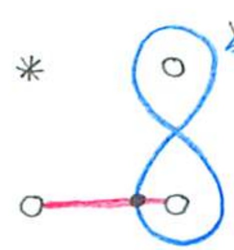
as complexes over k.

<u>def:</u> wrapped Lagrangian intersection Floer homology:

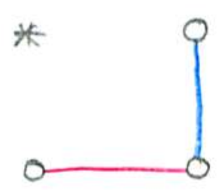
If L and L' are not parallel circles, then

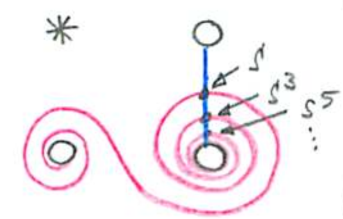
$$HF(L,L') \cong k^{\min \#(L \cap L') \text{ (up to wrapping)}}.$$

$$\text{eg} \quad H_*\Big(\mathrm{Mor}\big(\bullet, \; \circ \xrightarrow{H} \circ\big)\Big) \cong HF\big(\varphi(\bullet), \varphi(\circ \xrightarrow{H} \circ)\big)$$

$= HF\Bigg($  $\Bigg) = k$

$$\text{eg} \quad H_*\big(\mathrm{Mor}(\bullet, \circ)\big) = HF\big(\varphi(\bullet), \varphi(\circ)\big)$$

$\neq HF\Bigg($  $\Bigg) = 0$?

$= HF\Big($ 

$\Big) = k\langle s, s^3, s^5, \dots\rangle = k[H]$

<ins>so:</ins> $\boxed{H \cong \text{"wrapping around punctures"}}$

If L and L' are parallel circles with local systems X and X', respectively, then

$$HF(L, L') = k^{\min \#(L \cap L')} \oplus \left(k^2 \otimes \ker\left(X' \otimes X^{-1} - \mathrm{id}\right)\right).$$

<ins>cor:</ins> (Geometric Pairing Theorem)

$$\widetilde{BN}(\mathcal{L}(T_1, T_2)) \cong HF\left(\widehat{BN}(mT_1), \widehat{BN}(T_2)\right)$$

Also, if $\widetilde{Kh}(T) := \varphi\left(\underline{\Delta}^1(T)\right)$ then

$$\widetilde{Kh}(\mathcal{L}(T_1, T_2)) \cong HF\left(\widehat{BN}(mT_1), \widetilde{Kh}(T_2)\right)$$

<ins>Applications:</ins> Conway mutation

<ins>thm:</ins> [Bloom, Wehrli] $\widetilde{Kh}(L; \mathbb{F}_2)$ is mutation invariant.

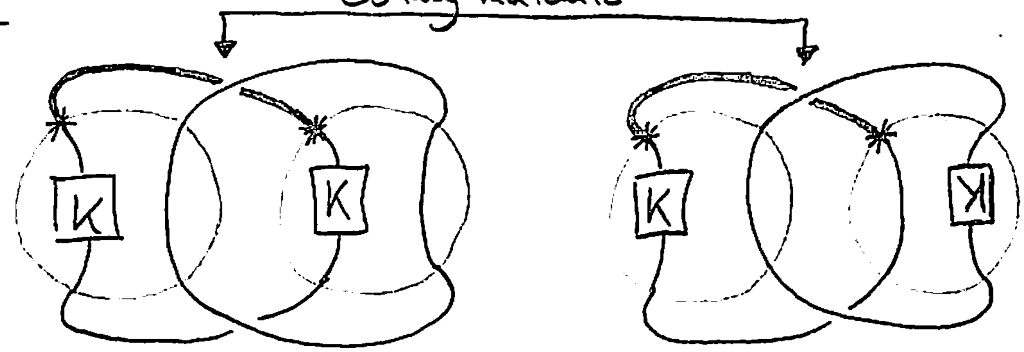
◀ Now, this follows from pairing + moving basepoint *. ▶

<ins>thm:</ins> [KWZ] $\widetilde{BN}(L; \mathbb{F}_2)$ is mutation invariant.

◀ same proof ▶

example:

Conway mutants

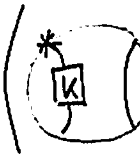
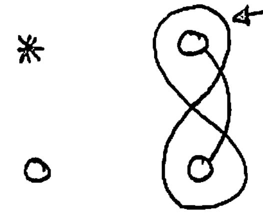


$(K\#K)_* \cup O$ $\qquad$ $K_* \cup K$

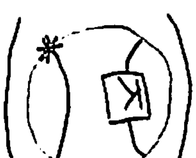
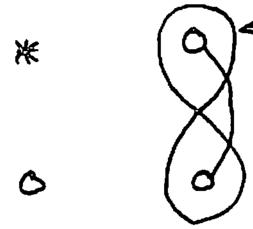
thm: [Wehrli] If $K = (2,n)$-torus link, then Kh of these links over $\mathbb{Q}$ is different.

example: Say $n=3$, ie K = trefoil. Then

local system $X = (-1)$

$L = \widetilde{BN}\Big($  $\Big) =$  $\hat{=} \Big(\circ \oplus \big(\circ \xrightarrow{SS'-D} \circ\big)\Big)$

local system $X' = (+1)$

$L' = \widetilde{BN}\Big($  $\Big) =$  $\hat{=} \Big(\circ \oplus \big(\circ \xrightarrow{S'S+D} \circ\big)\Big)$

So in $HF(L,L)$, the local systems contribute

$$k^2 \otimes \ker\big((-1)\otimes(-1)^{-1} - id\big) = k^2 \otimes \ker\big(\underbrace{(+1) - id}_{=0}\big) = k^2,$$

whereas in $HF(L,L')$, we see

$$k^2 \otimes \ker\big((+1)\otimes(-1)^{-1} - id\big) = k^2 \otimes \ker\big((-2)\big) = \begin{cases} k^2 & \text{if } \mathrm{char}(k) = 2 \\ 0 & \text{otherwise} \end{cases}.$$

observation: If $\widetilde{\mathrm{BN}}(T)$ does not contain circles (like $T = T_{2,-3}$, see (Q9)), mutating T preserves $\widetilde{\mathrm{BN}}$ and $\widetilde{\mathrm{Kh}}$ over any field.

thm: [Lee] For knot K, $\widetilde{\mathrm{BN}}(K;k) \cong \underbrace{\widetilde{\mathrm{BN}}^{\mathrm{free}}(K;k)}_{=k[H]} \oplus (H\text{-torsion})$

cor: If a tangle T has no closed components, then $\widetilde{\mathrm{BN}}(T)$ contains a single non-compact component $\mathrm{arc}(T)$.

◀ Pair $\widetilde{\mathrm{BN}}(T)$ with $\widetilde{\mathrm{BN}}(\ast)$. Then observe that $\mathrm{HF}(L, L')$ is finite-dimensional unless L and L' contain arcs ending at the same puncture: . ▶

cor: If $K = \mathcal{L}(T_1, T_2)$ is a knot, then

$$\widetilde{\mathrm{BN}}^{\mathrm{free}}(K;k) \subseteq \mathrm{HF}(\mathrm{arc}(T_1), \mathrm{arc}(T_2)).$$

def: Rasmussen's s-invariant

$$s^k(K) := \underline{\text{quantum grading}} \text{ of } 1 \in k[H] = \widetilde{\mathrm{BN}}^{\mathrm{free}}(K;k).$$

$\widetilde{\mathrm{Kh}}$ and $\widetilde{\mathrm{BN}}$ carry a bigrading:

- homological grading (vanishes on $\mathcal{B}$)
- quantum grading q $(q(D) = -2, q(S) = -1)$

cor: [KWZ] s^k is mutation invariant over any field k.

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