Perspectives on bordered Heegaard Floer homology

Exercises

Jake's lecture 1

(1) Show that $\mathrm{Sym}^n(S^2) = \mathbb{C}\mathrm{P}^n$.

(2) Consider the genus two Heegaard diagram for S^3 with 3 generators shown in class. Describe the branched covers of the disk for the two domains in question, and show they both contribute to the differential.

John's lecture 1

(3) Prove (say, via a Dehn surgery argument) that the open book with annular page and monodromy a single Dehn twist about its core supports S^3.

(4) Suppose L is braided about the unknot in S^3. The open book decomposition associated to the fibration of the unknot lifts to an open book decomposition of the branched double cover of S^3 along L. Determine the monodromy of the lifted open book in terms of the braid word.

(5) Suppose S is a genus one surface with one boundary component. Suppose h is the diffeomorphism of S given by a product $D_x D_y$ of positive Dehn twists about curves x and y in S which intersect in a single point. What is the fractional Dehn twist coefficient of h?

John's lecture 2

(6) Prove that any open book can be made right-veering after sufficient positive stabilizations.

(7) Prove that there is an exact triangle relating $\widehat{\mathrm{HF}}$ and HF^+ of the form:

$$\cdots \to \widehat{\mathrm{HF}} \to \mathrm{HF}^+ \xrightarrow{U} \mathrm{HF}^+ \to \widehat{\mathrm{HF}} \to \dots$$

Use this to show that if Y is a rational homology 3-sphere then Y is an L-space iff $\mathrm{HF}^+_{\mathrm{red}}(Y) = 0$.

Jake's lecture 2

(8) Let W be the cobordism obtained by attaching $Y_a \times I$, $Y_b \times I$ and $Y_c \times I$ to $\Delta \times \Sigma$. Fill in the details in the categorical argument showing that the map induced by W is given by composition.

(9) Consider the cobordism $W\colon S^3 \to S^3$ obtained by attaching a (-1)-framed handle along the unknot. Compute the maps $F^-_{W,s}$ directly in terms of holomorphic triangle counts.

Jonathan's lecture 1

(10) Draw Heegaard diagrams for p/q framed solid tori for several values of p/q and compute CFD from these diagrams. Write down CFA for these examples using the conversion algorithm described in lecture. For at least one pair, compute the box tensor product.

(11) Given a type D structure and a type A structure, show that the differential on the box tensor product satisfies $\partial^2 = 0$.

John's lecture 3

(12) Prove (say, via a Heegaard diagram argument) that stabilizing an open book does not change the manifold it supports.

(13) Prove that the universal Abelian cover of $L(p,q)\#L(q,p)$ is $\#^{2g}S^1 \times S^2$, where $2g = (p-1)(q-1)$.

Jake's lecture 3

(14) Consider the sutured manifold (M, γ), where M is the complement of the standard Seifert surface for $P(3,3,3)$ and γ is a longitude. Describe the effect of decomposing M along the 3 obvious disks. Compute SFH.

Jonathan's lecture 2

(15) For two of the solid tori CFDs computed earlier, write down the corresponding curve/train track. Reflect one and pair it with the other as in lecture (before pulling the curves tight). Identify each intersection point with a generator of the box tensor product and each bigon with a term in the differential of the box tensor product. (*Note:* This is harder/more interesting for some values of p and q than for others). Pulling the curves tight, guess the form of the curve for an arbitrary p/q framed solid torus.

(16) Suppose a curve segment of a train track ϑ_1 intersects two parallel curve segments of ϑ_2 that are connected by a crossover arrow. Show that if the segment of ϑ_1 is slid past the crossover arrow, the intersection Floer chain complex of ϑ_1 and ϑ_2 is modified by a change of basis (and thus the Floer homology is preserved).

(17) (a) Given a train track ϑ in the parametrized torus and two intersection points x and y, with the parametrizing curves (ie the sides of the square), show that adding a crossover arrow near the sides of the square from (the part of ϑ near) x to (the part of ϑ near) y changes the corresponding matrix by conjugating with an elementary matrix.

(b) *Challenge:* Prove that any train track coming from an extendable type D structure over the torus algebra is equivalent to one which has the form of immersed curves plus clockwise moving crossover arrows by proving the analogous matrices over $\mathbb{F}[U]/U^2$. That is, any $2n \times 2n$ matrix M over $\mathbb{F}[U]/U^2$ which is upper triangular mod U and squares to $U.I$ can be obtained from a matrix M' representing n arcs through the square by conjugating with the elementary matrices corresponding to clockwise moving crossover arrows.

Jonathan's lecture 3

(18) (a) Complete the proof outlined in lecture that two nontrivial, non-loose immersed curves with no peg-wrapping in the punctured torus intersect at least 4 times.

(b) Show that if two such curves intersect exactly 4 times, then either they are parallel or one of them corresponds to a non-primitive homology class in the torus.

(19) Show that if both sets of curves are allowed to pass through the basepoint, the intersection computes the size of $H_1(M_1 \cup M_2)$ rather than $\widehat{\mathrm{HF}}$. (*Hint:* In this setting, the curve set for M_i is homotopic to a copy of the homological longitude of M_i for each Spin^c structure of M_i.) Using this, explain why the Euler characteristic of $\widehat{\mathrm{HF}}(Y)$ is $|H_1(Y)|$ if Y is a rational homology sphere and 0 otherwise.