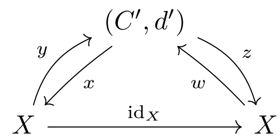
Exercises for "Immersed curves in Khovanov homology"

After lecture 1.

(1) Define notions of chain maps and homotopy equivalence for complexes over a category.

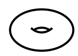
(2) (Cancellation Lemma) Suppose a complex (C,d) over a category $\mathcal{C}$ has the form



where X is an object in $\mathcal{C}$, C' is a direct sum of objects in $\mathcal{C}$ and d' is a matrix of morphisms from C' to C'. Find a differential d'' on C' (in terms of d', x, y, z and w) such that (C,d) is chain homotopic to (C', d''). Note that d'^2 is not necessarily 0.

(3) Using the Cancellation Lemma above, finish the proof of invariance of $[\![T]\!]_{/l}$ under the Reidemeister I move.

(4) In his definition of $\mathrm{Cob}_{/l}$, Bar-Natan also included the T-relation

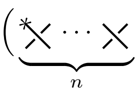
 $= 2.$

It says that whenever a cobordism contains a component which is a torus, the cobordism is equal to twice the same cobordism but without this torus component. Show that the T-relation follows from the S- and the $4Tu$-relations, unless the torus is the only component of the cobordism.

After lecture 2.

(5) Prove the neck-cutting relation. Also verify that the first pair of delooping isomorphisms from the cheat sheet agrees with the one from the lectures. Show that the two isomophisms are homotopic to those of the second pair.

(6) Show by induction or otherwise that

$Д\Big($  $\Big) = \Big(\bullet \xrightarrow{S} \circ \xrightarrow{D} \circ \xrightarrow{SS} \circ \xrightarrow{D} \cdots \longrightarrow \circ \Big)$

(7) What happens to the isomorphism of $\mathcal{B}$ with the full subcategory of $\mathrm{Cob}_{/l}$ when you change the basepoint $*$?

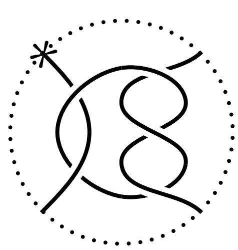
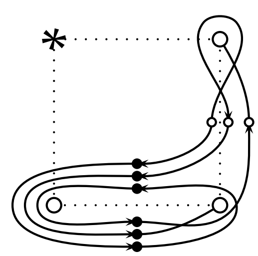
After lecture 3.

(8) (Clean-Up Lemma) Let (C,d) be a complex over a category $\mathcal{C}$ and suppose h is a matrix of (grading preserving) morphisms from C to C such that

$$h^2 \quad \text{and} \quad hdh$$

vanish. Show that (C,d) is isomorphic to $(X, d + dh - hd)$.

(9) Using the invariants of the 2- and 3-twist rational tangles, delooping as well as the Cancellation and Clean-Up Lemmas, verify that $\widetilde{\mathrm{BN}}(T_{2,-3})$ of the $(2,-3)$-pretzel tangle

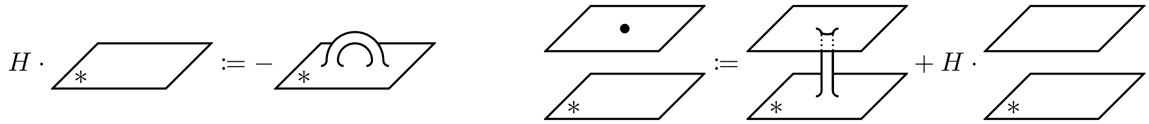
$T_{2,-3} \coloneqq$  is equal to 

After lecture 4.

(10) Using the result from exercise 9, compute $\widetilde{\mathrm{Kh}}(T_{2,-3})$. Calculate the reduced Bar-Natan and Khovanov homologies of the unknot and the $(2,5)$-, $(3,4)$- and $(3,5)$-torus knots.

Cheat sheet for “Immersed curves in Khovanov homology”

Notation for cobordisms with a distinguished component $*$.

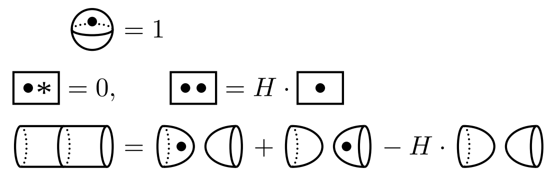


Induced relations.

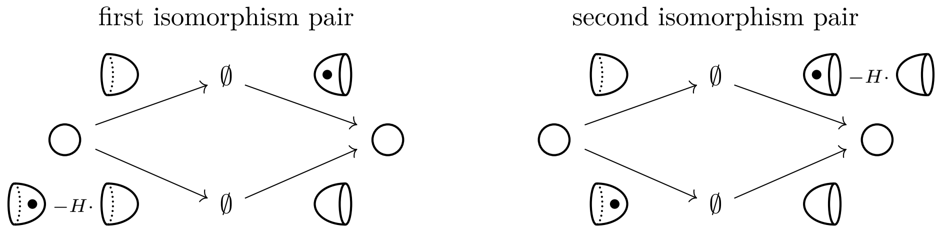
($S_\bullet$-relation)

(0- and H-trading relation)

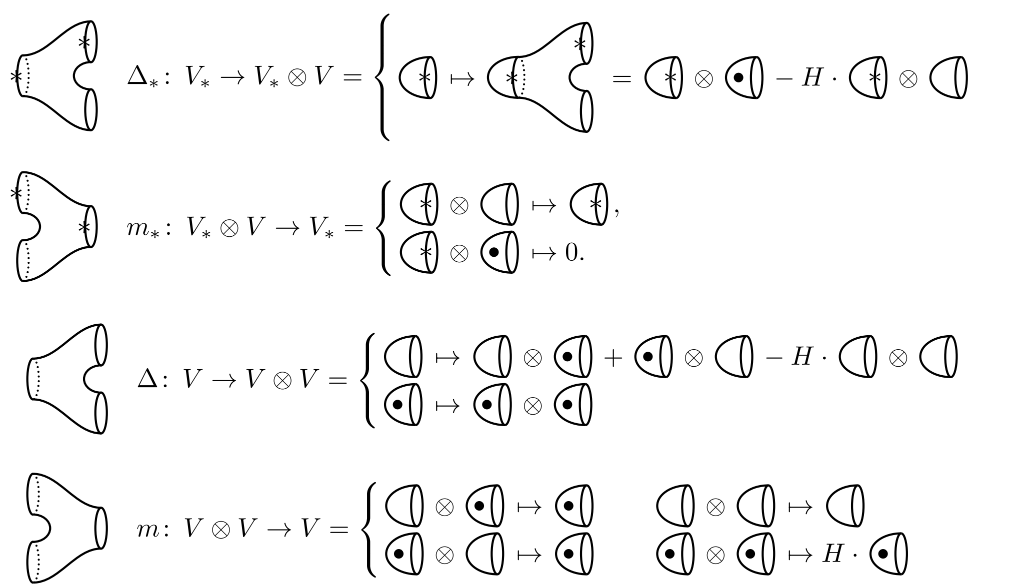
(Neck-cutting relation)



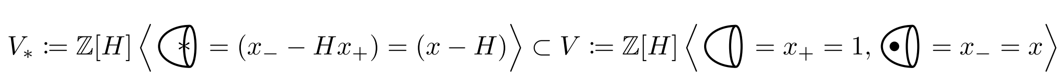
Delooping.



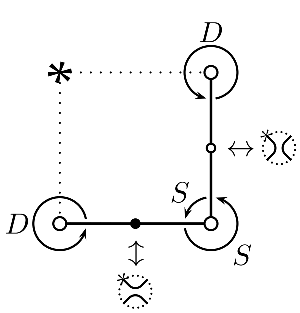
Split and merge maps in the algebraic definition of Bar-Natan homology.



where



The geometric interpretation of $\mathcal{B}$.



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