Khovanov homology and Conway mutation

joint with Artem Kotelskiy and Liam Watson

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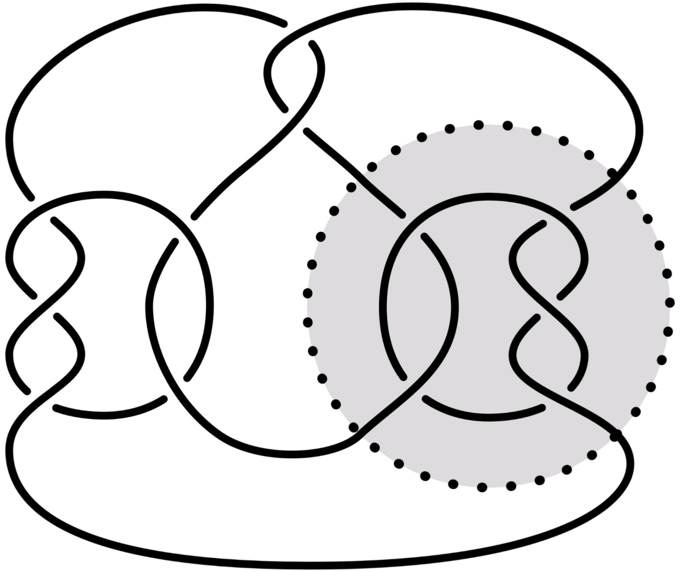
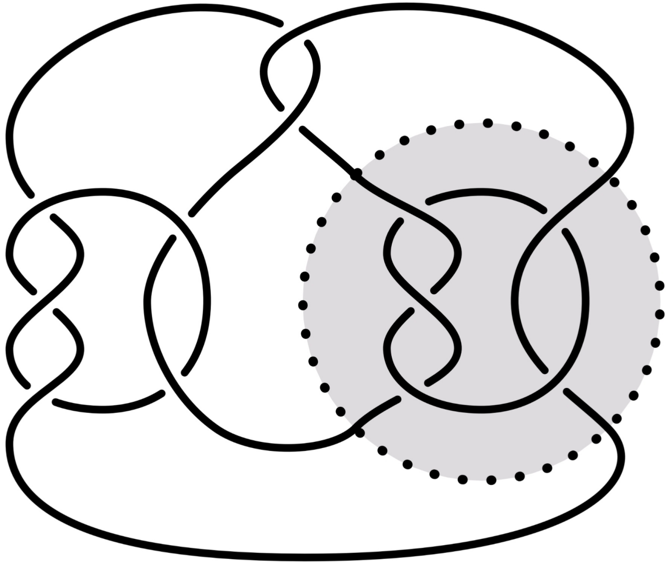
Conway mutation

[Conway '67]

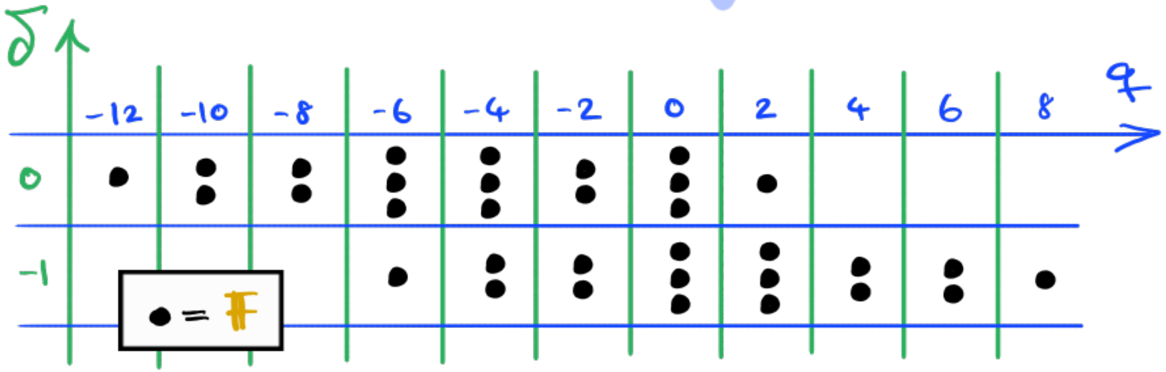


$L = T \cup T'$

$$\widetilde{Kh}(L) \cong \widetilde{Kh}(L')$$

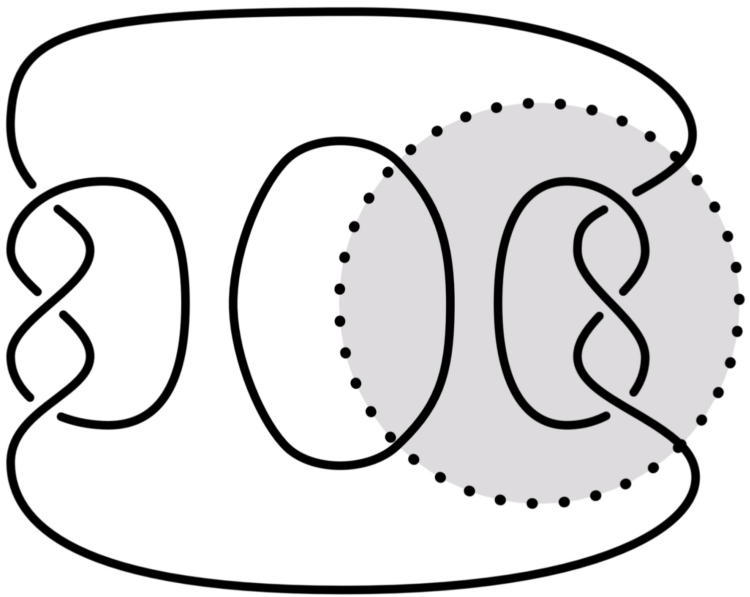


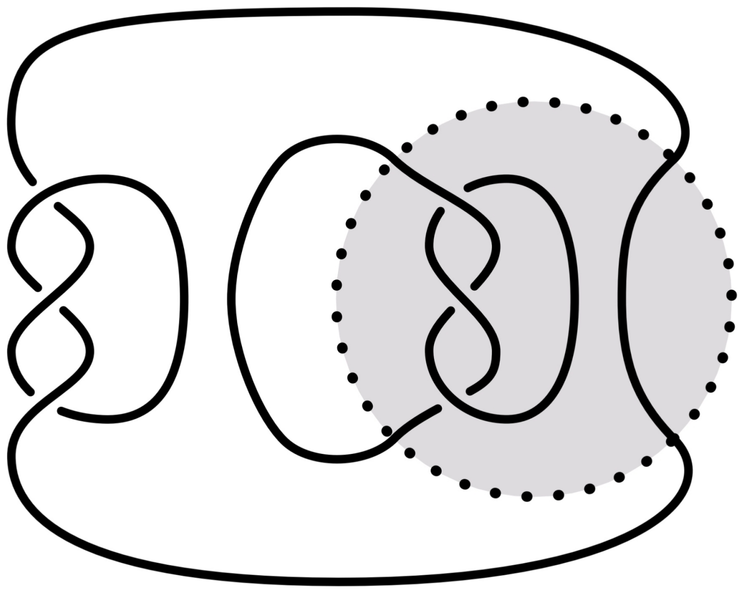
$L' = T \cup \mu(T')$



Example

[Wehrli '03]

$L =$ 

$L' =$ 

$$\dim_{\mathbb{Q}} \widehat{Kh}(L;\mathbb{Q}) \neq \dim_{\mathbb{Q}} \widehat{Kh}(L';\mathbb{Q})$$

$$\dim_{\mathbb{Z}/2\mathbb{Z}} \widehat{Kh}(L;\mathbb{Z}/2\mathbb{Z}) \overset{?}{=} \dim_{\mathbb{Z}/2\mathbb{Z}} \widehat{Kh}(L';\mathbb{Z}/2\mathbb{Z})$$

Theorem [Wehrli'10, Bloom'10]

For any two mutant links L and L',

$$\widetilde{Kh}(L;\mathbb{Z}/2\mathbb{Z}) \cong \widetilde{Kh}(L';\mathbb{Z}/2\mathbb{Z}).$$

Main Theorem [Kotelskiy-Watson-Z'26]

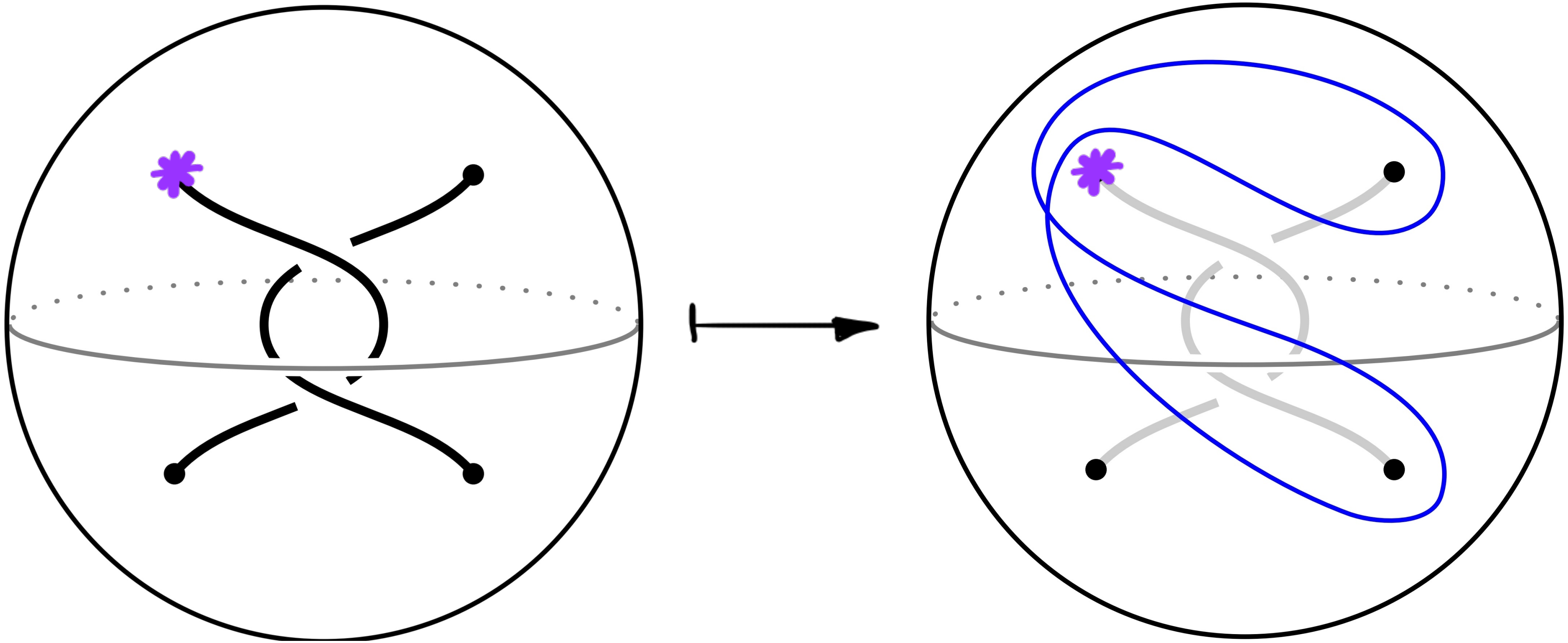
For any two mutant knots K and J and any field $\mathbb{F}$,

$$\widetilde{Kh}(K;\mathbb{F}) \cong \widetilde{Kh}(J;\mathbb{F}).$$

Khovanov multicurves

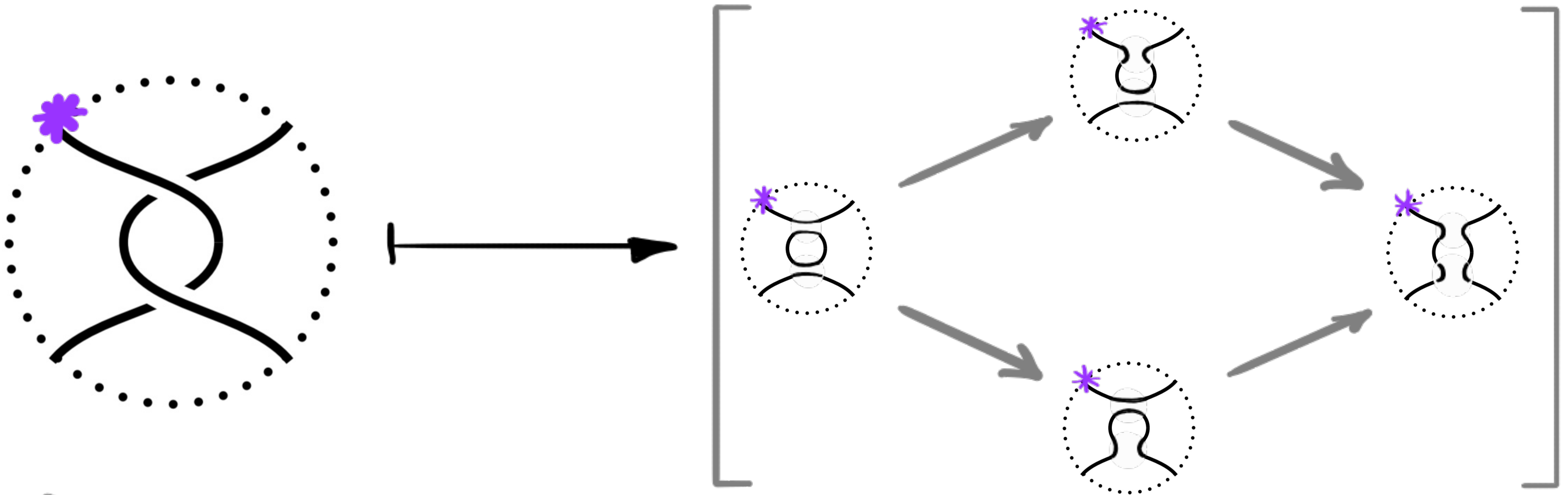
[KWZ'19]

pointed Conway tangle $T \subset B^3 \rightsquigarrow$ object $\widetilde{Kh}(T)$ in $\mathrm{Fuk}(D^2_3)$



Construction step 1 : Cube of resolutions

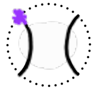
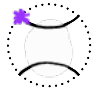
pointed Conway tangle diagram $D_T \subset B^3$ [Bar-Natan '05] f.g. bigraded chain complex $[\![D_T]\!]$ over $\mathrm{Cob}_{/l}$

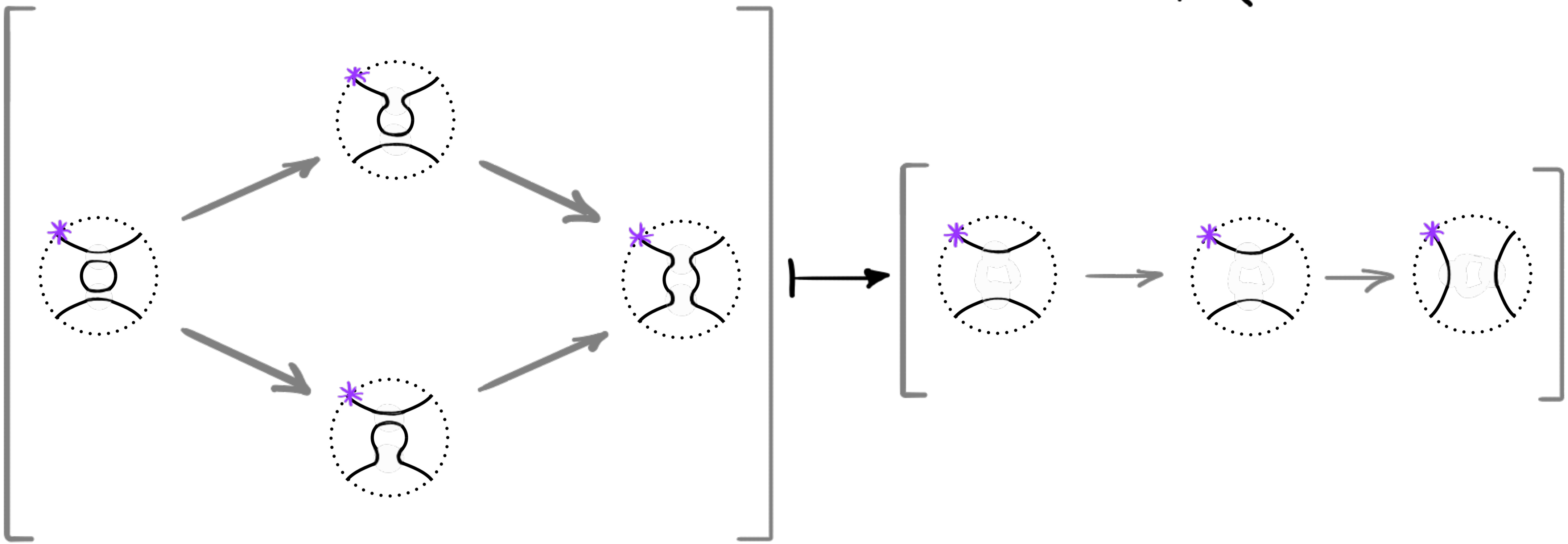


Theorem [Bar-Natan '05]

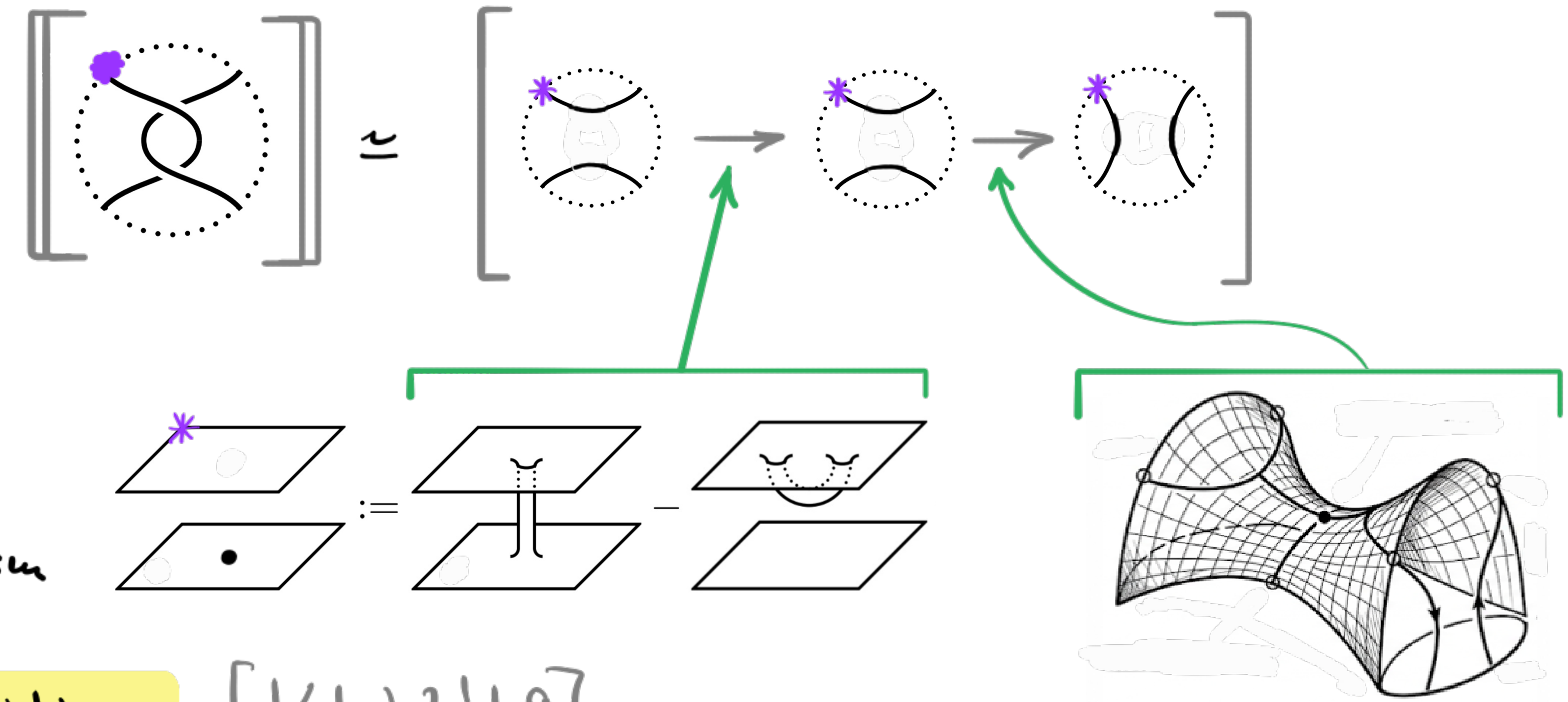
$[\![T]\!] := [\![D_T]\!]$ is an invariant up to chain homotopy equivalence.

Construction step 2: Delooping

f.g. bigraded chain complex $[\![T]\!]$ over $\mathrm{Cob}_{/\ell}$ $\xrightarrow{\text{[Bar-Natan'05]}}$ f.g. bigraded chain complex $[\![T]\!]$ over $\mathrm{Cob}_{/\ell}$( $\oplus$ )



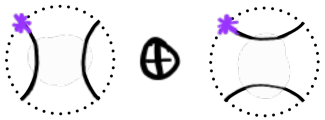
Understanding the category $\mathrm{Cob}_{/\ell}$



dot cobordism

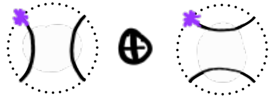
saddle cobordism*

Proposition [KWZ'19]

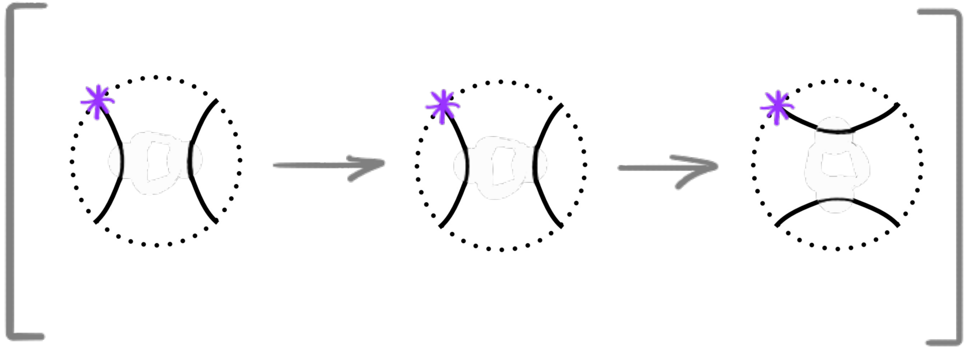
$\mathrm{Cob}_{/\ell}\Big($  $\Big) \cong \mathcal{B} := \mathbb{F}\left[\begin{tikzcd} \circ \arrow[loop left, "D_\circ"] \arrow[r, bend right, "S_\circ"'] & \bullet \arrow[l, bend right, "S_\bullet"'] \arrow[loop right, "D_\bullet"] \end{tikzcd}\right] \Big/ \begin{matrix} SD=0 \\ DS=0 \end{matrix}$

* as imagined by Gemini 3.1 Pro

Construction step 3: Apply isomorphism

f.g. bigraded chain complex over $\mathrm{Cob}_{/l}\Big($  $\Big)$ $\rightsquigarrow$ f.g. bigraded chain complex over $\mathcal{B}$

$$[\![T]\!] \longmapsto \Delta(T)^{\mathcal{B}}$$



$\longmapsto \quad \left[\, \circ \xrightarrow{D_0} \circ \xrightarrow{S_0} \bullet \,\right]$

Construction step 4: Take mapping cone

f.g. bigraded chain complex over B $\rightsquigarrow$ f.g. bigraded chain complex over B

$$\Delta(T)^B \longmapsto \Delta_1(T)^B := \mathrm{Cone}\left(G \cdot 1_{\Delta(T)^B}\right)$$

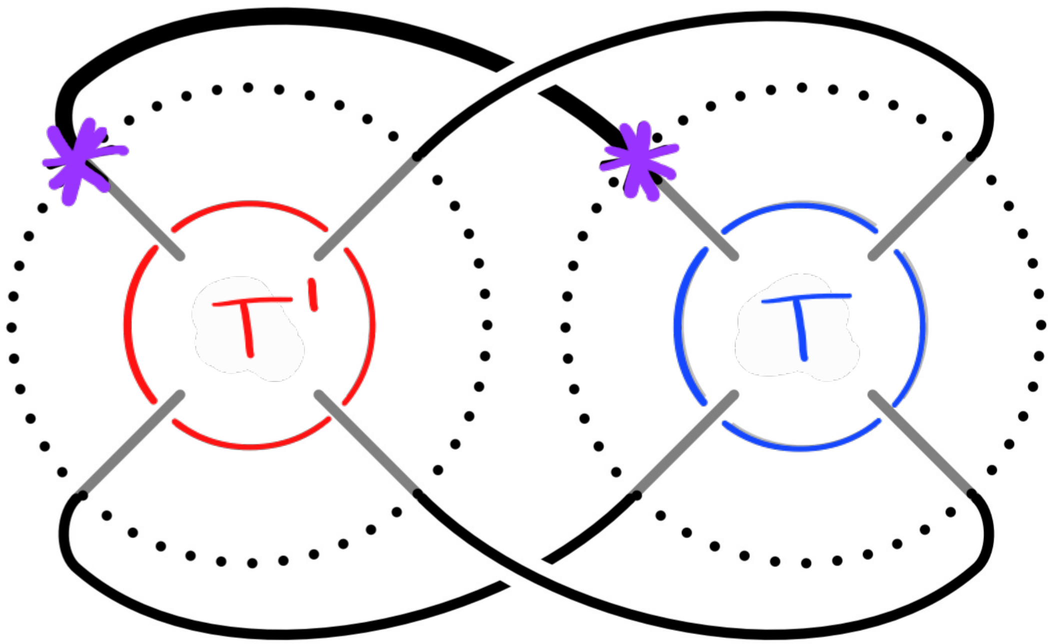
$$G := S_\circ S_\bullet + S_\bullet S_\circ - D_\bullet - D_\circ$$

$$\left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right] \longmapsto \left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] \arrow[d, "G"'] & \circ \arrow[r, "S_\circ"] \arrow[d, "G"'] & \bullet \arrow[d, "G"'] \\ \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right]$$

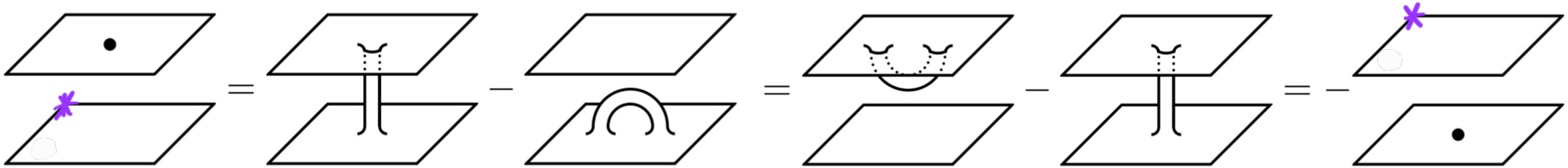
Proposition (Algebraic Glueing Theorem)

[Bar-Natan'05, Manion'17, KWZ'19]

$$\widetilde{Kh}(T' \cup T) \cong H_*\left(\mathrm{Mor}\left(-\Delta_1(T')^B, \Delta(T)^B\right)\right)$$



Dependence of $\Delta(T)^{\mathcal{B}}$ on the basepoint $*$



Corollary [KWZ'19]

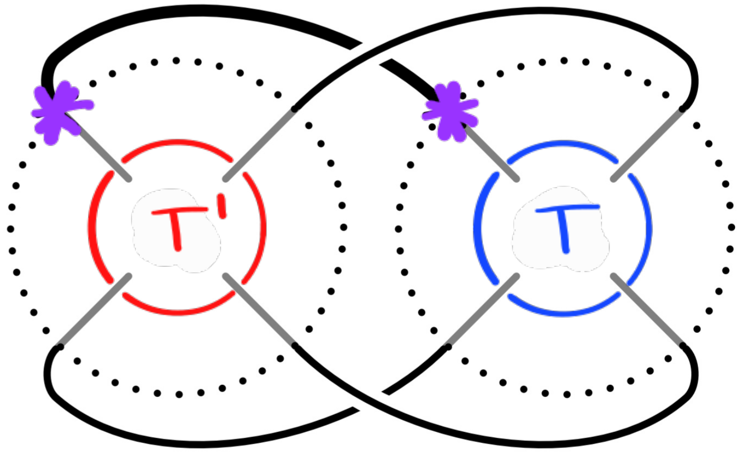
For any pointed Conway tangle T,

$$\Delta(\mu(T))^{\mathcal{B}} = \mu\left(\Delta(T)^{\mathcal{B}}\right)$$

where $\mu\colon \mathcal{B} \to \mathcal{B}$ is the isomorphism

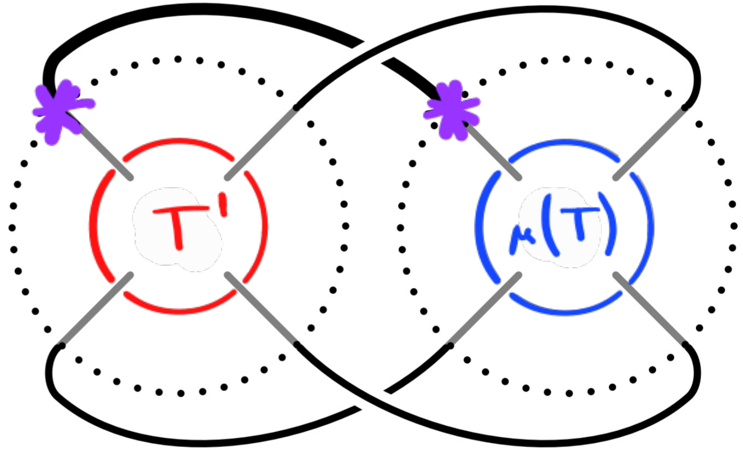
$$S_\bullet \mapsto S_\bullet,\quad S_\circ \mapsto S_\circ,\quad D_\bullet \mapsto (-D_\bullet),\quad D_\circ \mapsto (-D_\circ)$$

Conway mutation and reduced Khovanov homology



$$\widetilde{Kh}(T' \cup T)$$
$$\cong H_*\left(\mathrm{Mor}\left(-\Delta_1(T'),\ \Delta(T)\right)\right)$$

$\overset{?}{\cong}$



$$\widetilde{Kh}(T' \cup \mu(T))$$
$$\cong H_*\left(\mathrm{Mor}\left(-\Delta_1(T'), \Delta(\mu(T))\right)\right)$$
$$\cong H_*\left(\mathrm{Mor}\left(-\Delta_1(T'), \mu(\Delta(T))\right)\right)$$

Immediate consequences over $\mathbb{Z}/2\mathbb{Z}$

Corollary [KWZ'19]

For any pointed Conway tangle T,

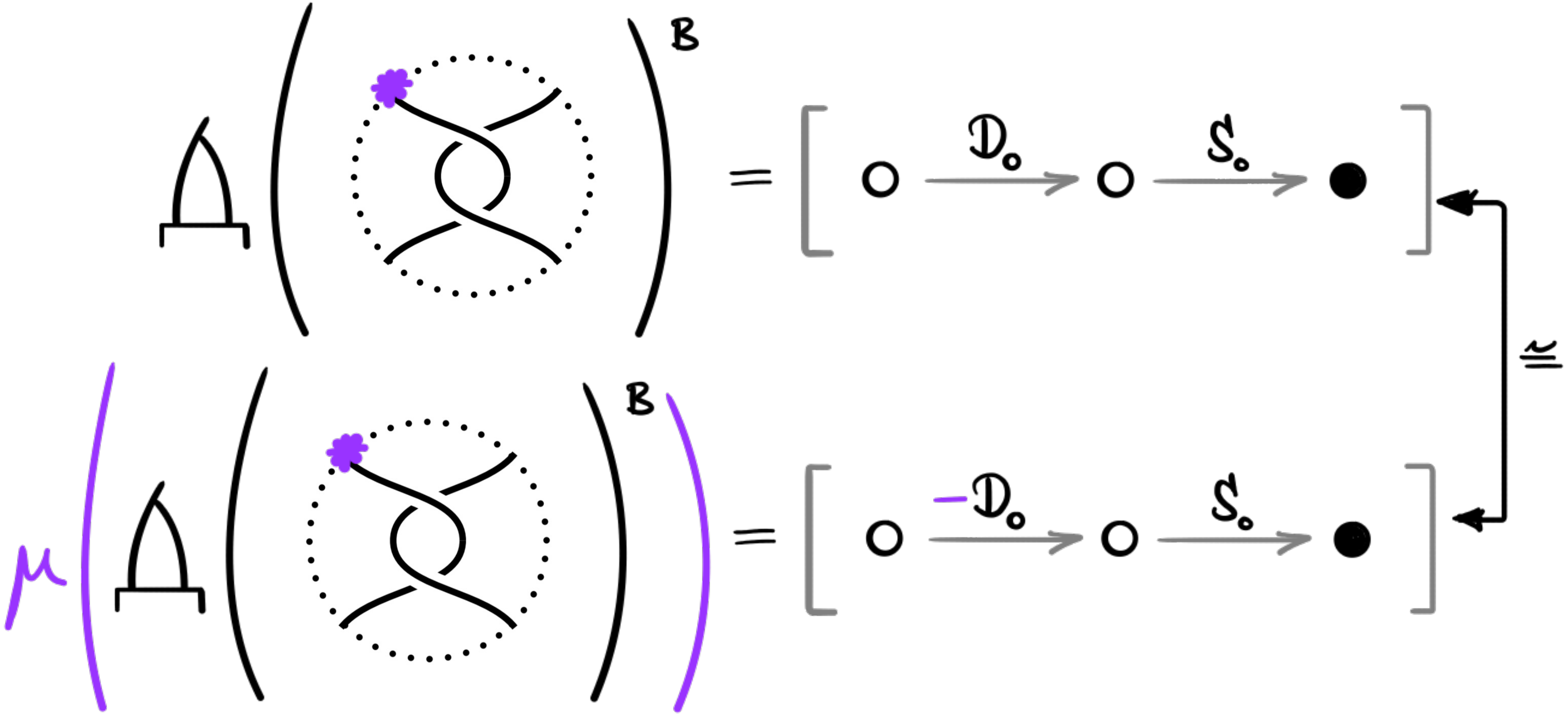
$$\Delta(\mu(T);\mathbb{Z}/2\mathbb{Z})^B = \Delta(T;\mathbb{Z}/2\mathbb{Z})^B.$$

Theorem [Wehrli'10, Bloom'10, KWZ'19]

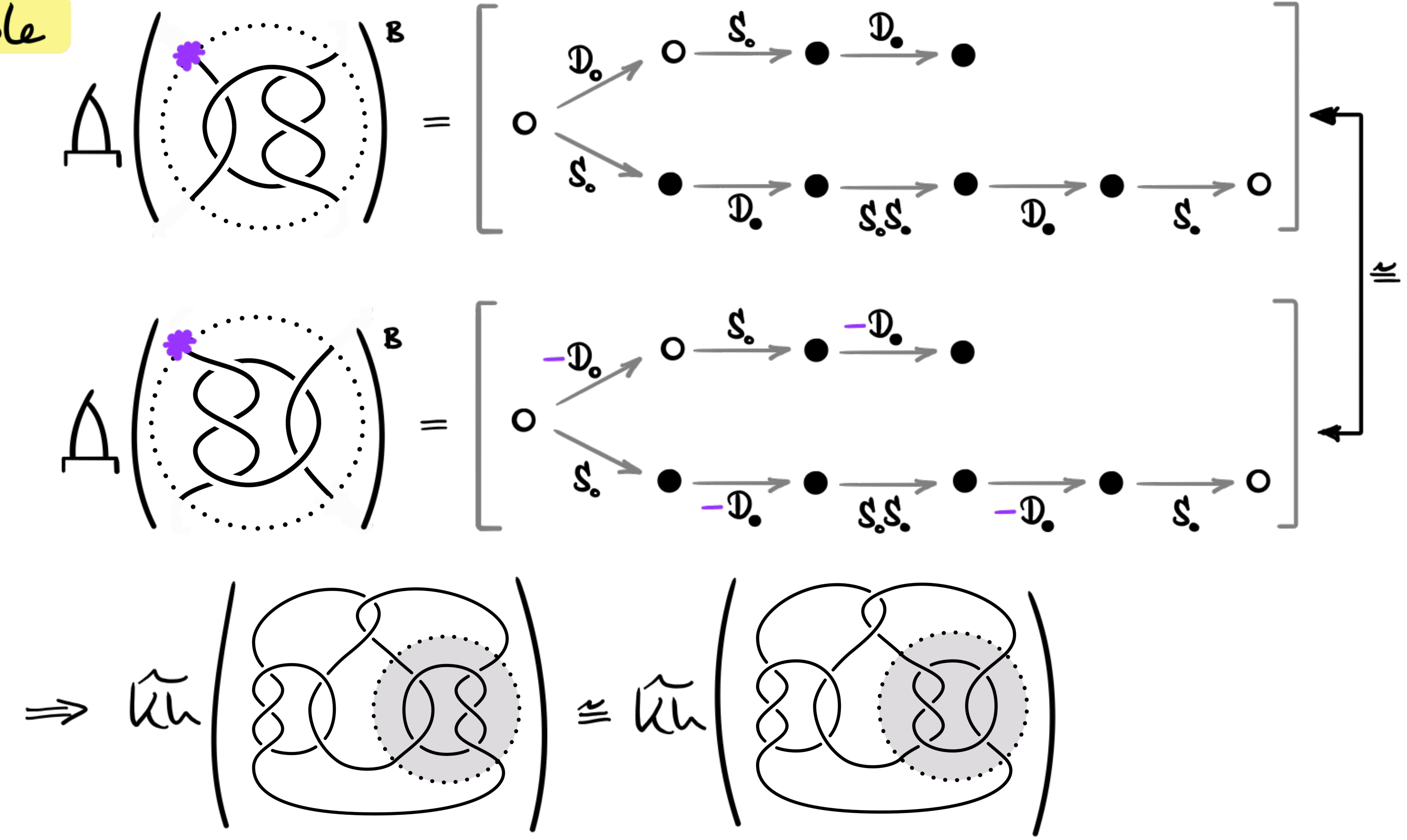
For any two mutant links L and L',

$$\widetilde{Kh}(L;\mathbb{Z}/2\mathbb{Z}) \cong \widetilde{Kh}(L';\mathbb{Z}/2\mathbb{Z}).$$

example

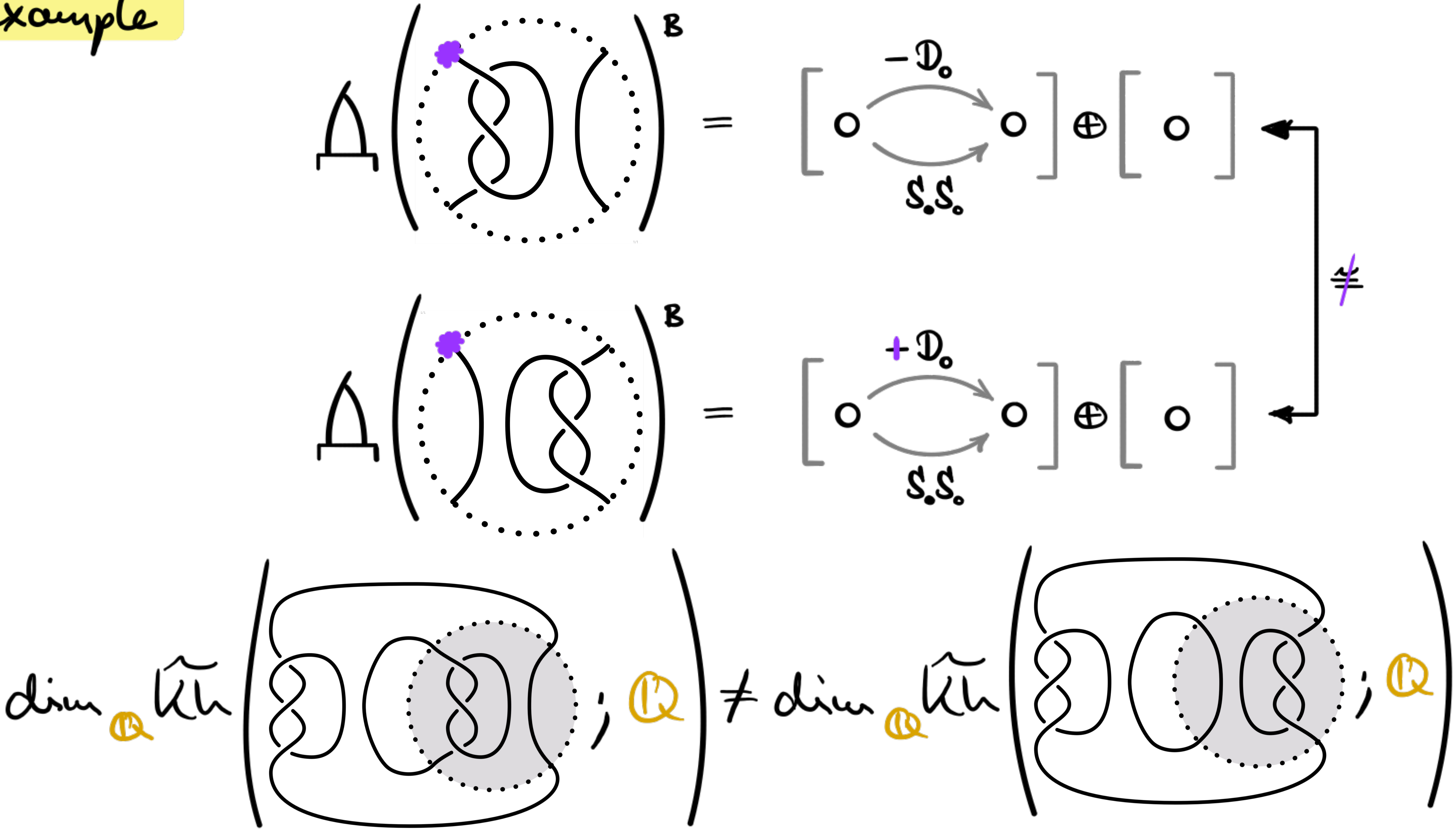


example



B
B
$D_\circ$ $S_\circ$ $D_\bullet$
$S_\circ$ $D_\bullet$ $S_\circ S_\bullet$ $D_\bullet$ $S_\bullet$
$-D_\circ$ $S_\circ$ $-D_\bullet$
$S_\circ$ $-D_\bullet$ $S_\circ S_\bullet$ $-D_\bullet$ $S_\bullet$
$\cong$
$\Rightarrow \widehat{Kh}$ $\cong$ $\widehat{Kh}$

example

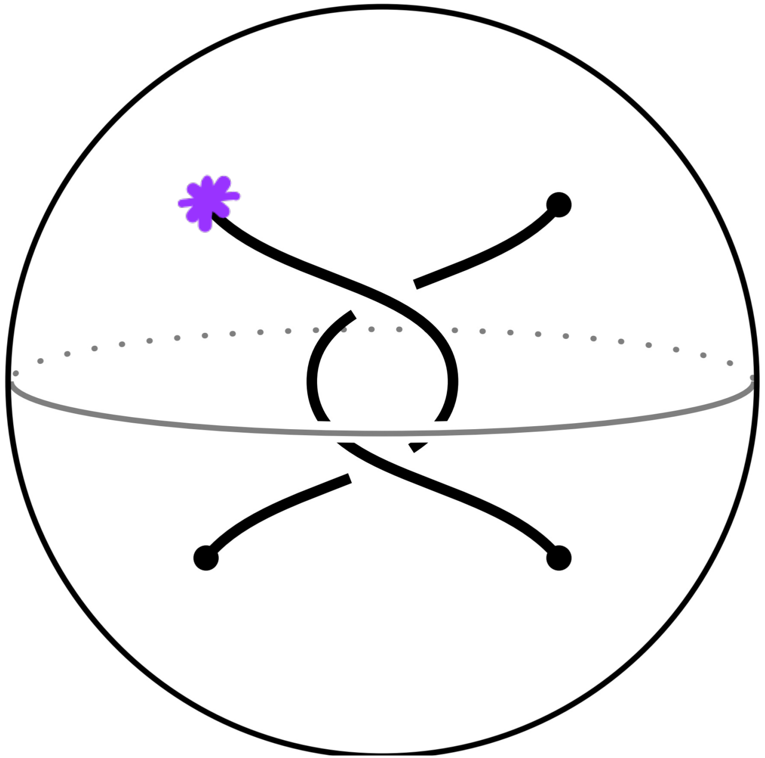


B
$-D_0$
$S_\bullet S_0$
$\oplus$
B
$+D_0$
$S_\bullet S_0$
$\oplus$
$\neq$
$\dim_{\mathbb{Q}} \widehat{Kh}$
; $\mathbb{Q}$
$\neq \dim_{\mathbb{Q}} \widehat{Kh}$
; $\mathbb{Q}$

Summary of constuction steps 1–4:

pointed Conway tangle $T \subset B^3$ $\rightsquigarrow$ [KWZ'19] f.g. bigraded chain complex over $\mathcal{B}$

$T =$



$\mapsto$

$$\Delta(T)^{\mathcal{B}} = \left[\ \circ \xrightarrow{D_\circ} \circ \xrightarrow{S_\circ} \bullet\ \right]$$

$\mapsto$

$$\Delta_1(T)^{\mathcal{B}} = \left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] \arrow[d, "G"'] & \circ \arrow[r, "S_\circ"] \arrow[d, "G"'] & \bullet \arrow[d, "G"'] \\ \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right]$$

Fact from higher representation theory

Every chain complex over B is chain homotopic to a direct sum of cyclic and linear chain complexes over B.

e.g. $\left[\ \circ \xrightarrow{D_\circ} \circ \xrightarrow{S_\circ} \bullet\ \right]$ and $[\ \circ\ ]$ ⟵ always mutation invariant

e.g. $\left[\begin{tikzcd} \circ \arrow[r, bend left, "D_\circ"] \arrow[r, bend right, "S_\bullet S_\circ"'] & \circ \end{tikzcd}\right]$ and $\left[\begin{tikzcd} \circ \arrow[r, "S_\circ"] \arrow[d, "D_\circ"'] & \bullet \arrow[d, "D_\bullet"] \\ \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right]$ ⟵ mutation invariant $\Updownarrow$ $(\# D_\circ + \# D_\bullet)$ is even

Construction step 5: Clean up

f.g. bigraded chain complex over B $\rightsquigarrow$ cyclic and linear chain complexes over B

$$\Delta(T)^B = \left[\ \circ \xrightarrow{D_\circ} \circ \xrightarrow{S_\circ} \bullet\ \right] \longmapsto \Delta^c(T)^B = \left[\ \circ \xrightarrow{D_\circ} \circ \xrightarrow{S_\circ} \bullet\ \right]$$

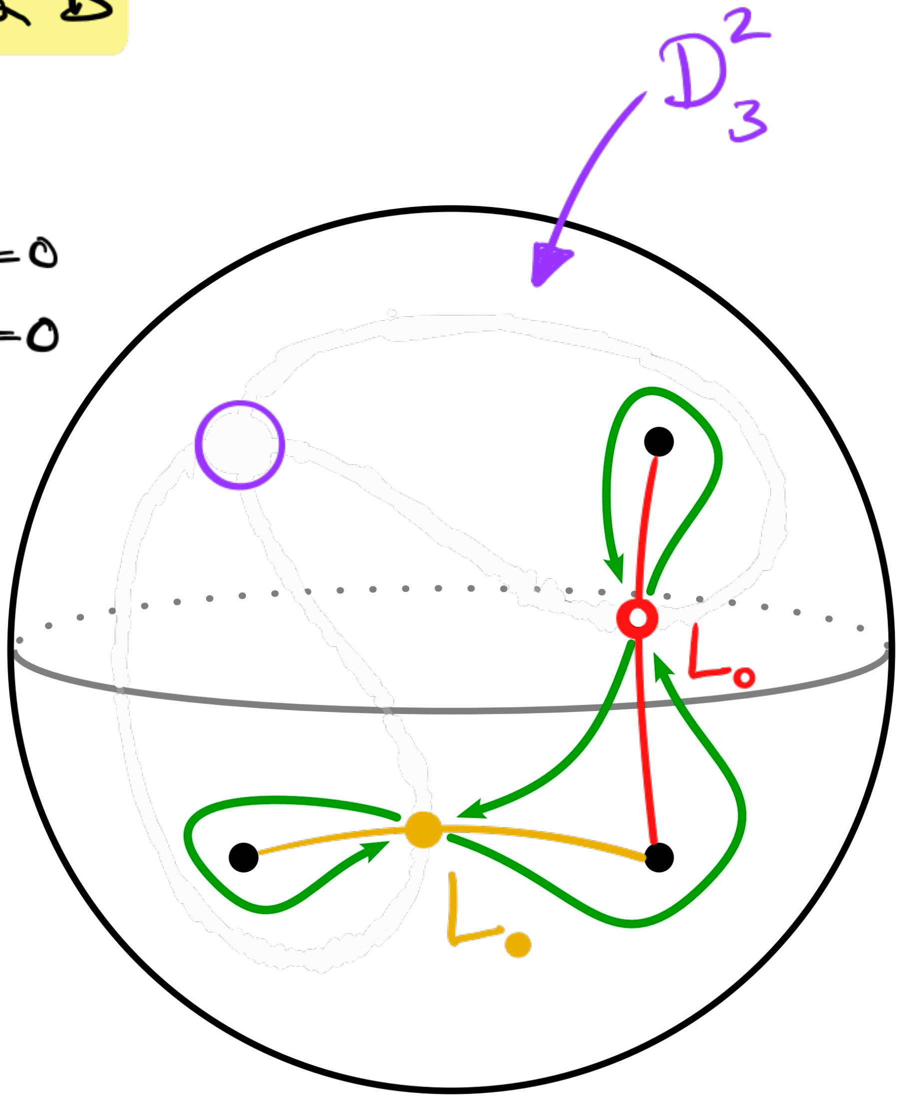
$$\Delta_1(T)^B = \left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] \arrow[d, "G"'] & \circ \arrow[r, "S_\circ"] \arrow[d, "G"'] & \bullet \arrow[d, "G"'] \\ \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right] \longmapsto \Delta_1^c(T)^B = \left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] \arrow[d, "S_\bullet S_\circ"'] & \circ \arrow[r, "S_\circ"] & \bullet \arrow[d, "-D_\bullet"'] \\ \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right]$$

Symplectic perspective on the algebra $\mathbb{B}$

$$\mathbb{B} = \mathbb{F}\left[\begin{tikzcd} \bullet \arrow[loop left, "D_\bullet"] \arrow[r, bend right, "S_\bullet"'] & \circ \arrow[l, bend right, "S_\circ"'] \arrow[loop right, "D_\circ"] \end{tikzcd}\right] \Big/ \begin{matrix} SD = 0 \\ DS = 0 \end{matrix}$$

$$\cong \mathrm{End}_{\mathcal{W}(D^2_3)}(L_\bullet \oplus L_\circ)$$

wrapped Fukaya category

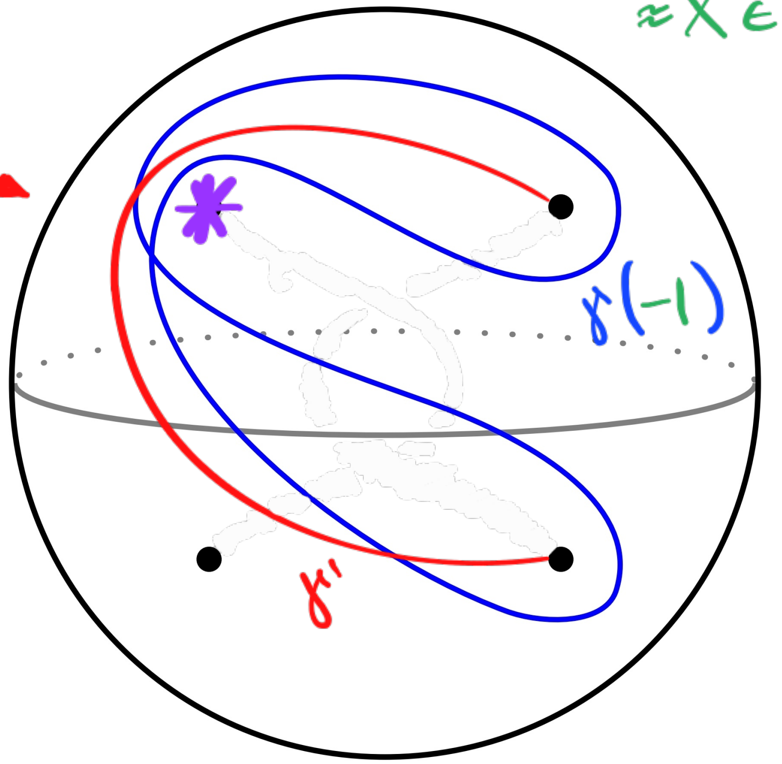


Theorem [Haiden-Katzarkov-Kontsevich'15, Hanselman-Rasmussen-Watson'16]

$$\frac{\left\{\text{(graded) chain complexes over } \mathcal{B}\right\}}{\text{chain homotopy}} \xleftrightarrow{\;1:1\;} \left\{\begin{array}{c}\text{(graded) multicurves on } D^2_3 \\ \text{with } \underbrace{\text{local systems}}_{\approx X \in GL_n(\mathbb{F})}\end{array}\right\}\Big/\sim$$

$$\left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right]$$

$$\left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] \arrow[d, "S_\bullet S_\circ"'] & \circ \arrow[r, "S_\circ"] & \bullet \arrow[d, "-D_\bullet"] \\ \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right]$$

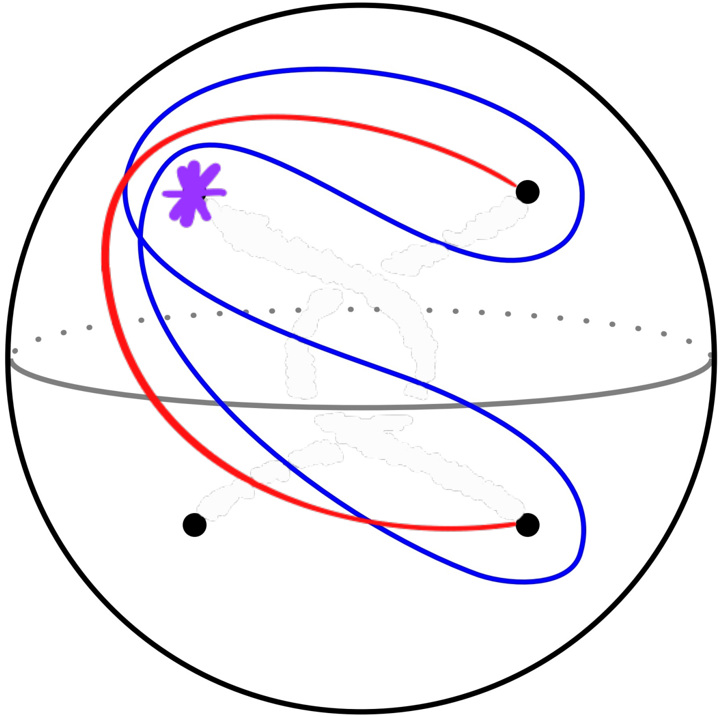


Construction step 6: Use correspondence

cyclic and linear chain complexes over B $\rightsquigarrow$ objects in $\mathrm{Fuk}(D^2_3)$

$$\Delta^c(T)^B = \left[\circ \xrightarrow{D_\circ} \circ \xrightarrow{S_\circ} \bullet\right] \longmapsto \widetilde{BN}(T)$$

$$\Delta^c_1(T)^B = \left[\begin{tikzcd} \circ \arrow[r, "D_\circ"] \arrow[d, "S_\bullet S_\circ"'] & \circ \arrow[r, "S_\circ"] & \bullet \arrow[d, "-D_\bullet"] \\ \circ \arrow[r, "D_\circ"] & \circ \arrow[r, "S_\circ"] & \bullet \end{tikzcd}\right] \longmapsto \widetilde{Kh}(T)$$

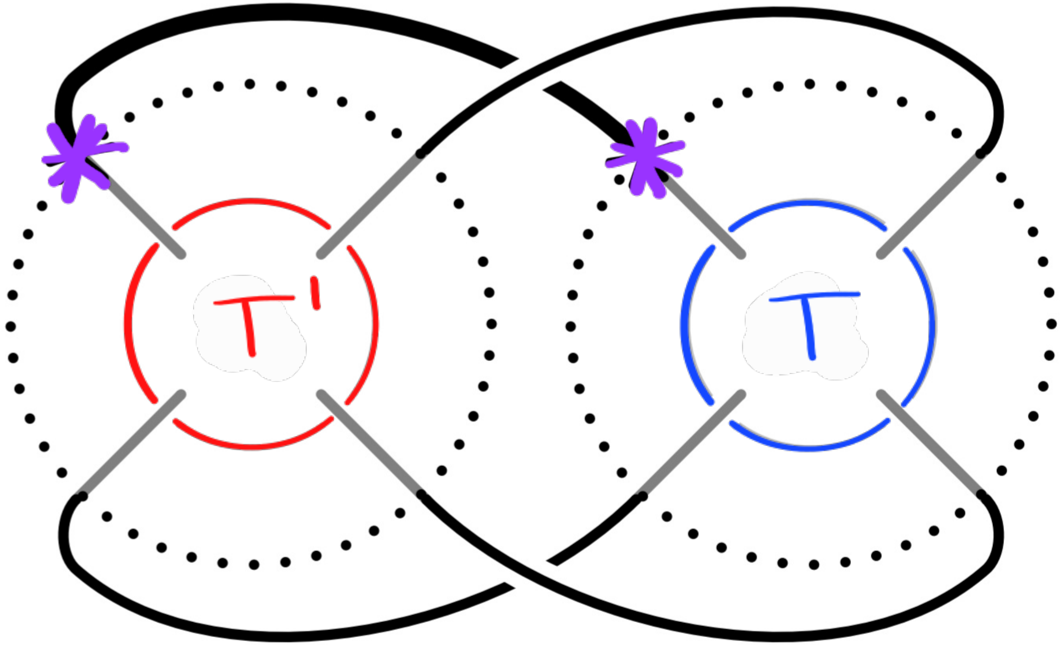


Mutation $\widetilde{BN}(\mu(T)) = \widetilde{BN}(T)$ up to multiplying local systems by (± 1).

Theorem (Geometric Glueing) [KWZ'19]

$$\widetilde{Kh}(T' \cup T) \cong HF\left(-\widetilde{Kh}(T'), \widetilde{BN}(T)\right)$$

Lagrangian Floer homology

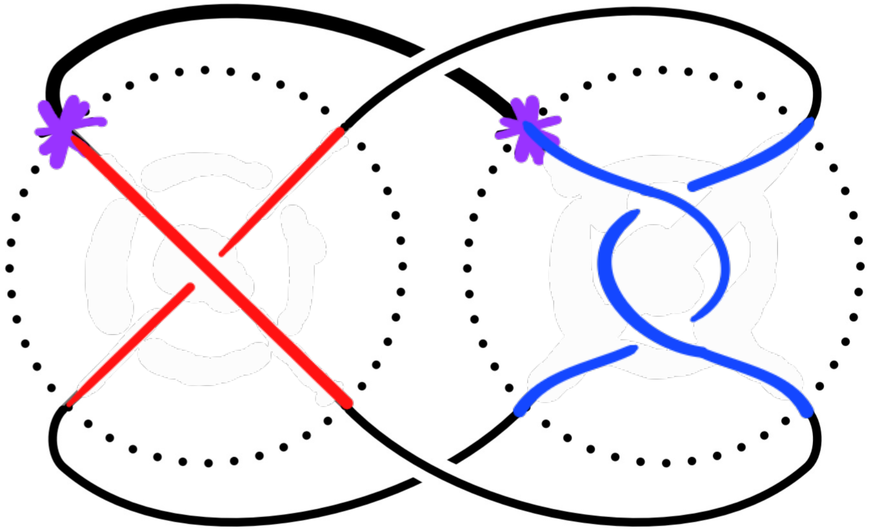


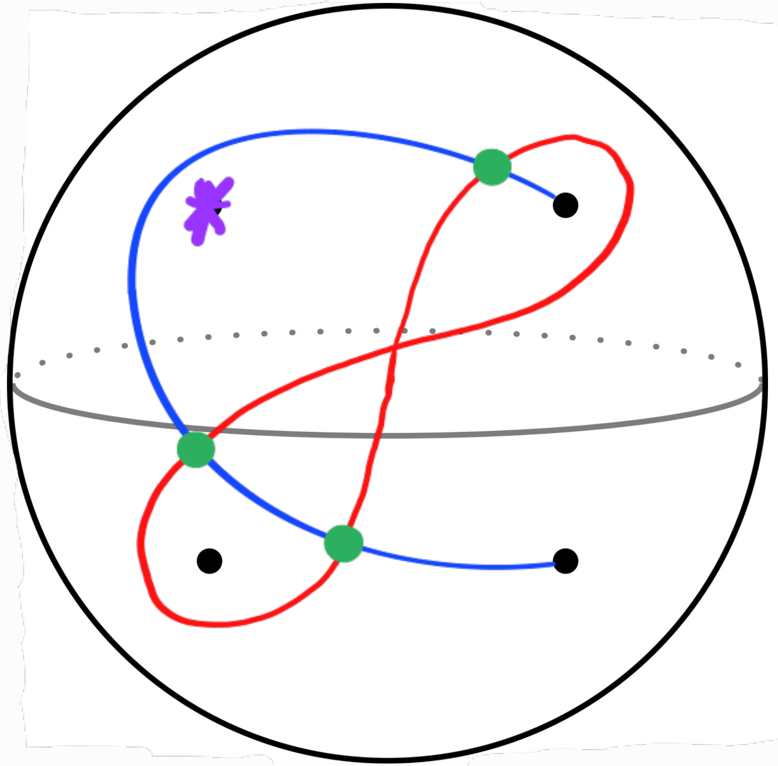
Theorem (Geometric Glueing) [KWZ'19]

$$\widetilde{Kh}(T' \cup T) \cong HF\left(-\widetilde{Kh}(T'), \widetilde{BN}(T)\right)$$

↖ Lagrangian Floer homology

example





Proof of the Main Theorem

$$K = T' \cup T \implies \widehat{Kh}(K;\mathbb{F}) \cong HF\left(-\widehat{Kh}(T'), \widetilde{BN}(T)\right)$$

$$J = T' \cup \mu(T) \implies \widehat{Kh}(J;\mathbb{F}) \cong HF\left(-\widehat{Kh}(T'), \widetilde{BN}(\mu(T))\right)$$

local system

$$\widetilde{BN}(T) = \coprod_{i=1}^{n} \gamma_i(X_i) \implies \widetilde{BN}(\mu(T)) = \coprod_{i=1}^{n} \gamma_i(\varepsilon_i \cdot X_i) \text{ for some } \varepsilon_i \in \{\pm 1\}$$

$\implies$ It suffices to show

$$HF(\gamma', \gamma_i(X_i)) \cong HF(\gamma', \gamma_i(\varepsilon_i \cdot X_i))$$

$\forall$ components γ' of $-\widehat{Kh}(T')$ and $\forall\, i \in \{1,\dots,n\}$

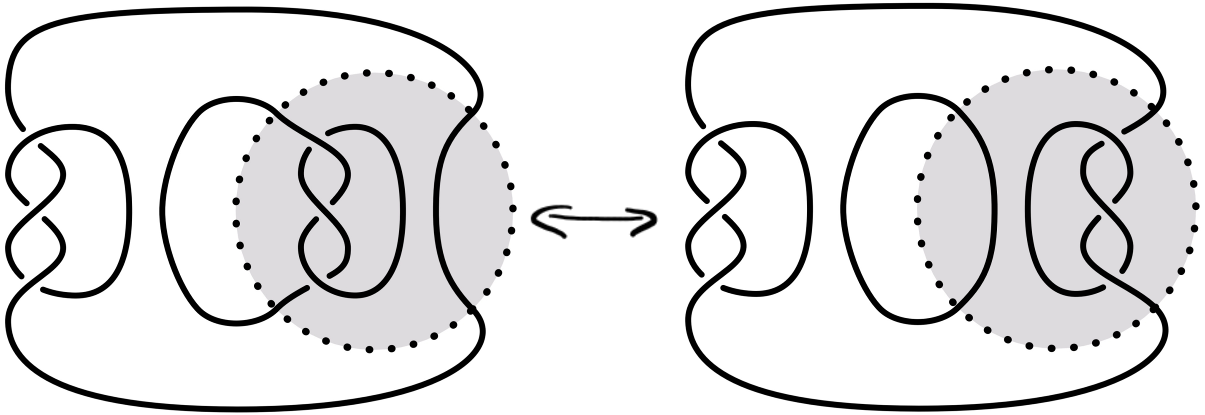
It suffices to show

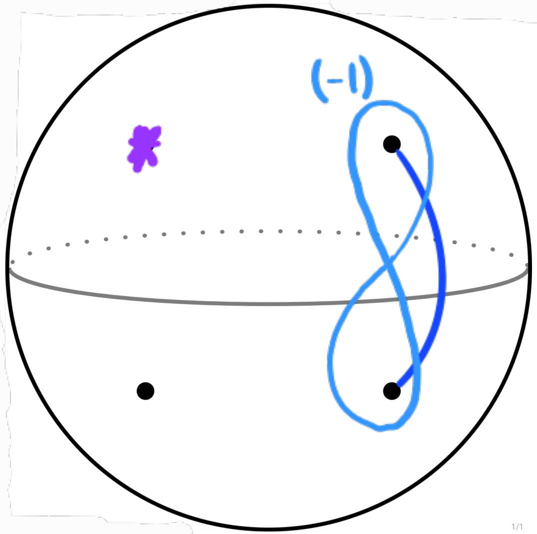
$$HF(\gamma', \gamma_i(X_i)) \cong HF(\gamma', \gamma_i(\varepsilon_i \cdot X_i))$$

$\forall$ components γ' of $-\widetilde{Kh}(T')$ and $\forall\, i \in \{1, \dots, n\}$

Case 1: $\gamma' \not\sim \gamma_i$.

$$\Rightarrow HF(\gamma', \gamma_i(X)) \cong HF(\gamma', \gamma_i(Y)) \quad \forall\, X, Y \in GL_n(\mathbb{F})$$





It suffices to show

$$HF(\gamma', \gamma_i(X_i)) \cong HF(\gamma', \gamma_i(\varepsilon_i \cdot X_i))$$

$\forall$ components γ' of $-\widetilde{Kh}(T')$ and $\forall\, i \in \{1, \dots, n\}$

Case 1: $\gamma' \not\simeq \gamma_i$. ✓ Case 2: $\gamma' \simeq \gamma_i$. ?

Theorem (Geography of $\widetilde{Kh}$) [KWZ'26]

For any Conway tangle T

$$\widetilde{Kh}(T) = \coprod_{i=1}^{n} r_{\ell_i}(p_i/q_i) \sqcup \coprod_{j=1}^{m} s_{2\ell'_j}(p'_j/q'_j)$$

"rational" "special"

where $\ell_i, \ell'_j \in \mathbb{Z}^{>0}$ and $p_i/q_i, p'_j/q'_j \in \mathbb{QP}^1 = \mathbb{Q} \cup \{\infty\}$.

Theorem (Rational tangle detection) [KWZ'19]

$$T = Q_{p/q} \iff \widetilde{Kh}(T) = r_1(p/q)$$

rational tangle of slope p/q

Theorem (Connectivity detection) [KWZ'26]

$$\left.\begin{array}{l} r_\ell(p/q) \in \widetilde{Kh}(T) \text{ or } \widetilde{BN}(T) \\ \quad \text{where } \ell \in \mathbb{Z}^{>0} \text{ is odd} \end{array}\right\} \implies x(T) = x(Q_{p/q}).$$

with any local system

Observation

$(\#D_\circ + \#D_\bullet)$ for $r_{2\ell}(p/q)$ and $s_{2\ell}(p/q)$ is even.

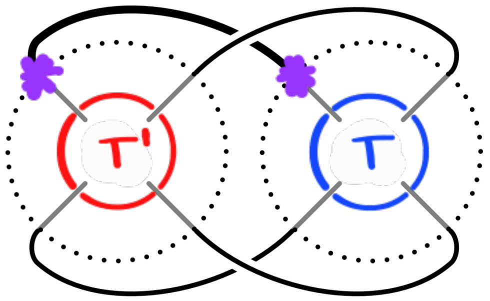
$$\boxed{\text{It suffices to show } \quad HF(\gamma', \gamma_i(X_i)) \cong HF(\gamma', \gamma_i(\varepsilon_i \cdot X_i)) \quad \forall \text{ components } \gamma' \text{ of } -\widetilde{Kh}(T') \text{ and } \forall i \in \{1, \dots, n\}}$$

Case 1: $\gamma' \not\simeq \gamma_i$. ✓ Case 2: $\gamma' \simeq \gamma_i$.

Geography of $\widetilde{Kh}$ $\Rightarrow$ $\gamma_i = r_\ell(p/q)$ or $s_{2\ell}(p/q)$

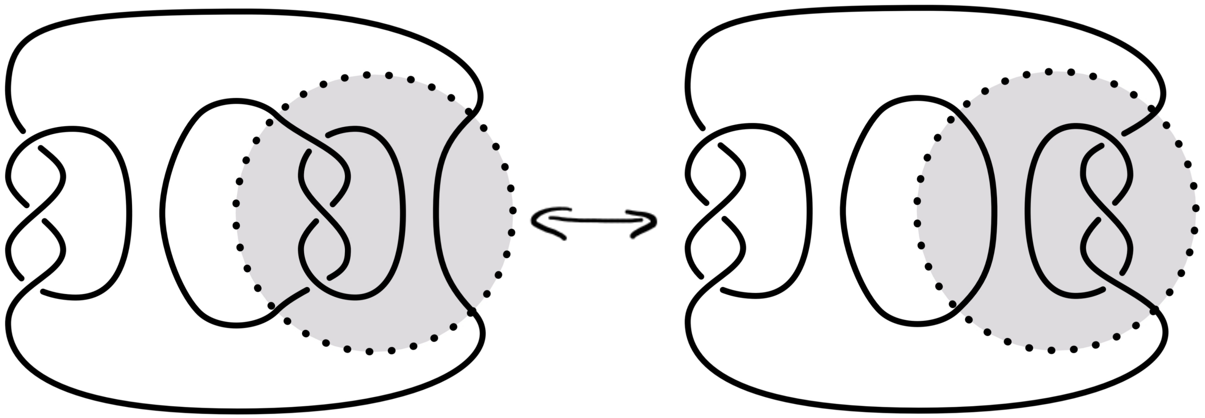
$x(T) \neq x(T')$ $\Rightarrow$ $\gamma_i \neq r_\ell(p/q)$ for odd ℓ

$\Rightarrow$ $\gamma_i = r_{2\ell}(p/q)$ or $s_{2\ell}(p/q)$

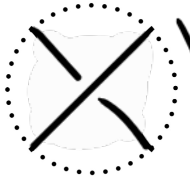


observation $\Rightarrow$ $\varepsilon_i = +1$. ∎

not component-preserving Conway mutation



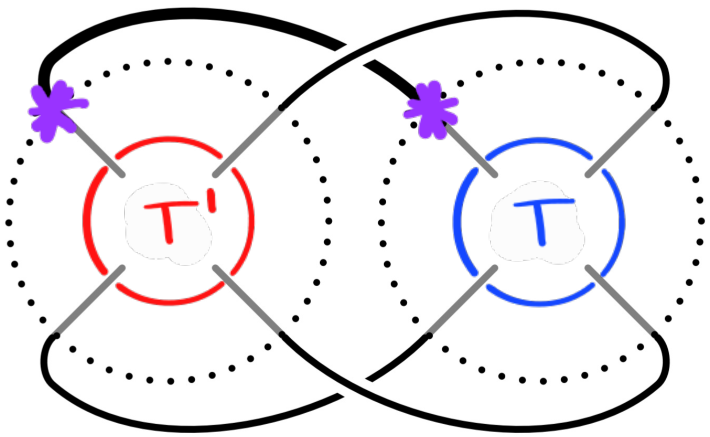
Observation

$(\#D_\circ + \#D_\bullet)$ for $r_e(p/q)$ is even if $x(Q_{p/q}) = x($  $)$.

Corollary [KWZ'26]

For any two links L and L' that are related by component-preserving Conway mutation and for any field $\mathbb{F}$,

$$\widehat{Kh}(L;\mathbb{F}) \cong \widehat{Kh}(L';\mathbb{F}).$$



Theorem (Geography of $\widetilde{Kh}$) [KWZ'26]

For any Conway tangle T

$$\widetilde{Kh}(T) = \coprod_{i=1}^{n} \underset{\text{"rational"}}{r_{\ell_i}}\left(p_i/q_i\right) \sqcup \coprod_{j=1}^{m} \underset{\text{"special"}}{s_{2\ell'_j}}\left(p'_j/q'_j\right)$$

where $\ell_i, \ell'_j \in \mathbb{Z}^{>0}$ and $p_i/q_i, p'_j/q'_j \in \mathbb{QP}^1 = \mathbb{Q} \cup \{\infty\}$.

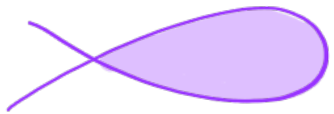
Symplectic perspective on the algebra $\mathbb{B}$

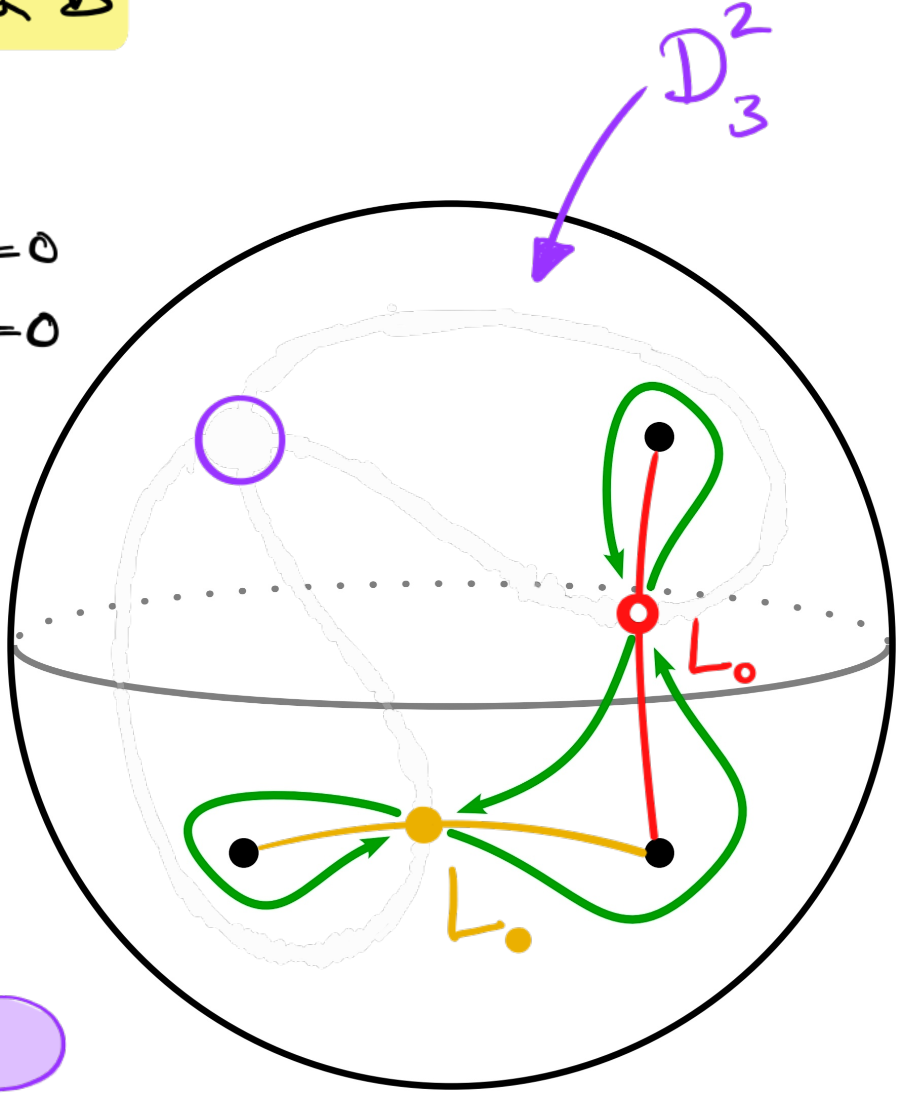
$$\mathbb{B} := \mathbb{F}\left[\begin{tikzcd} \bullet \arrow[loop left, "D_\bullet"] \arrow[r, bend right, "S_\bullet"'] & \circ \arrow[l, bend right, "S_\circ"'] \arrow[loop right, "D_\circ"] \end{tikzcd}\right] \Big/ \begin{matrix} SD=0 \\ DS=0 \end{matrix}$$

$$\cong \mathrm{End}_{\mathcal{W}(D^2_3)}(L_\bullet \oplus L_\circ)$$

wrapped Fukaya category

objects ≈ immersed curves w/o "teardrops"





An A_∞-structure on the algebra $\mathbb{B}$

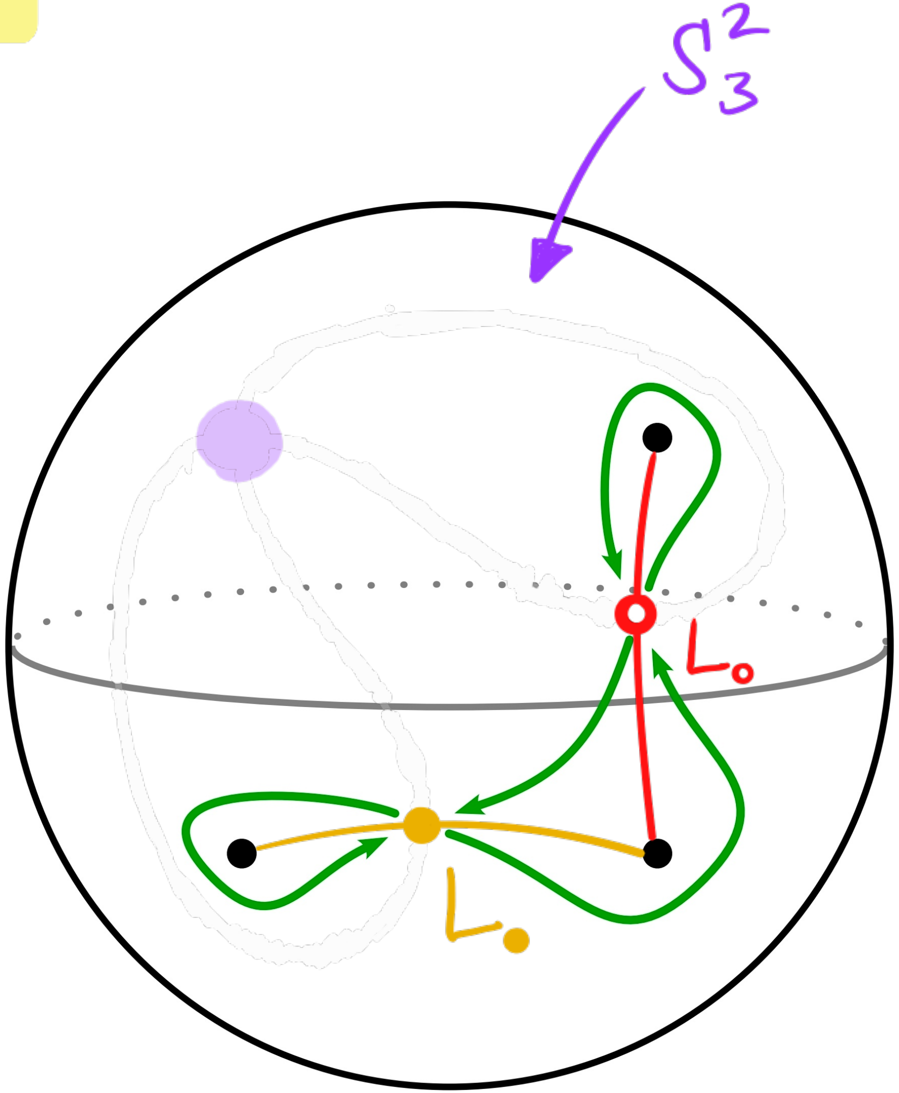
$$\mathbb{B} := \mathbb{F}\left[\begin{tikzcd} \bullet \arrow[loop left, "D_\bullet"] \arrow[r, bend right, "S_\bullet"'] & \circ \arrow[l, bend right, "S_\circ"'] \arrow[loop right, "D_\circ"] \end{tikzcd} \right]$$

$$B^\infty = \mathrm{End}_{\mathcal{W}(S^2_3)}(L_\bullet \oplus L_\circ)$$

$\mu_n : B^{\otimes n} \longrightarrow B$, $n \geq 2$,

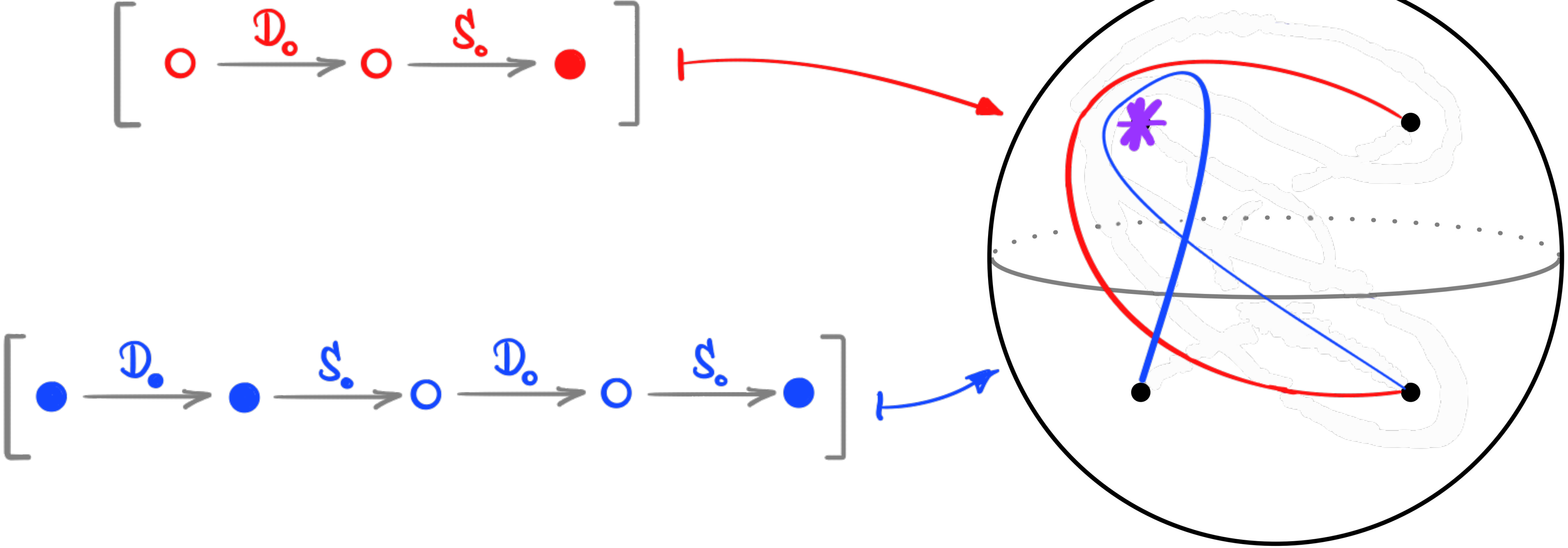
e.g. $\mu_4(D, S, D, S) = 1$,

and $\mu_4(S, D, S, D) = 1$



Theorem [Haiden-Katzarkov-Kontsevich '15, Hanselman-Rasmussen-Watson '16]

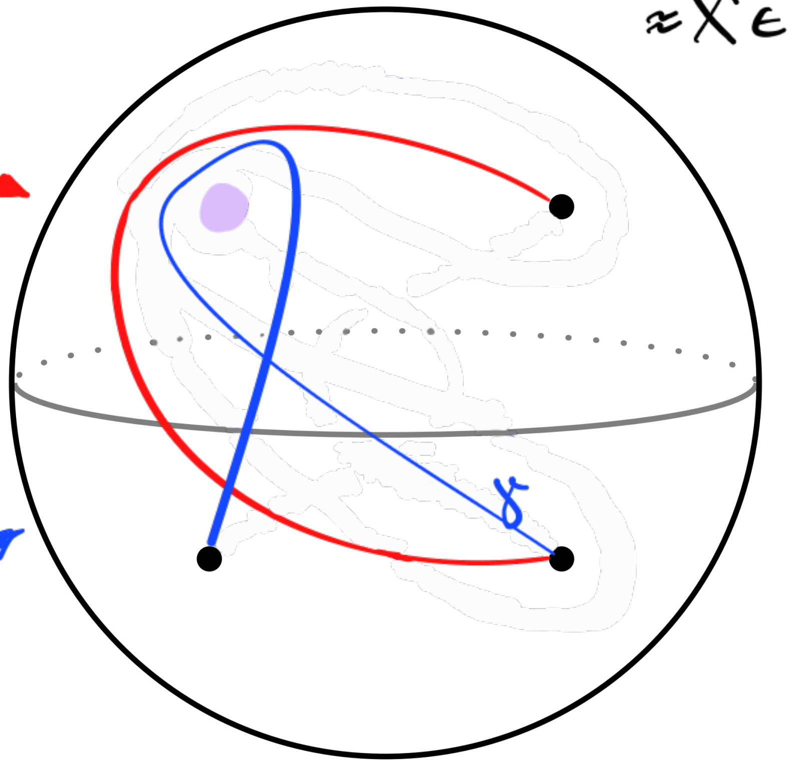
$$\frac{\{\text{(graded) chain complexes over } \mathcal{B}\}}{\text{chain homotopy}} \xleftrightarrow{1:1} \left\{ \begin{array}{c}\text{(graded) multicurves on } D^2_3 \\ \text{with } \underbrace{\text{local systems}}_{\approx X \in GL_n(\mathbb{F})}\end{array}\right\} \Big/ \sim$$



Theorem [Haiden-Katzarkov-Kontsevich'15, Hanselman-Rasmussen-Watson'16]

$$\frac{\{\text{(graded) chain complexes over } \mathcal{B}^\infty\}}{\text{chain homotopy}} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{(graded) multicurves on } S^2_3 \\ \text{with } \underbrace{\text{local systems}}_{\approx X \in GL_n(\mathbb{F})}\end{array}\right\}\Big/\sim$$

$$\left[\circ \xrightarrow{D_\circ} \circ \xrightarrow{S_\circ} \bullet\right] \longmapsto$$



$$\left[\bullet \xrightarrow{D_\bullet} \bullet \xrightarrow{S_\bullet} \circ \xrightarrow{D_\circ} \circ \xrightarrow{S_\circ} \bullet\right] \longmapsto$$

$\gamma \in \mathrm{Fuk}(D^2_3)$, but $\gamma \notin \mathrm{Fuk}(S^2_3)$

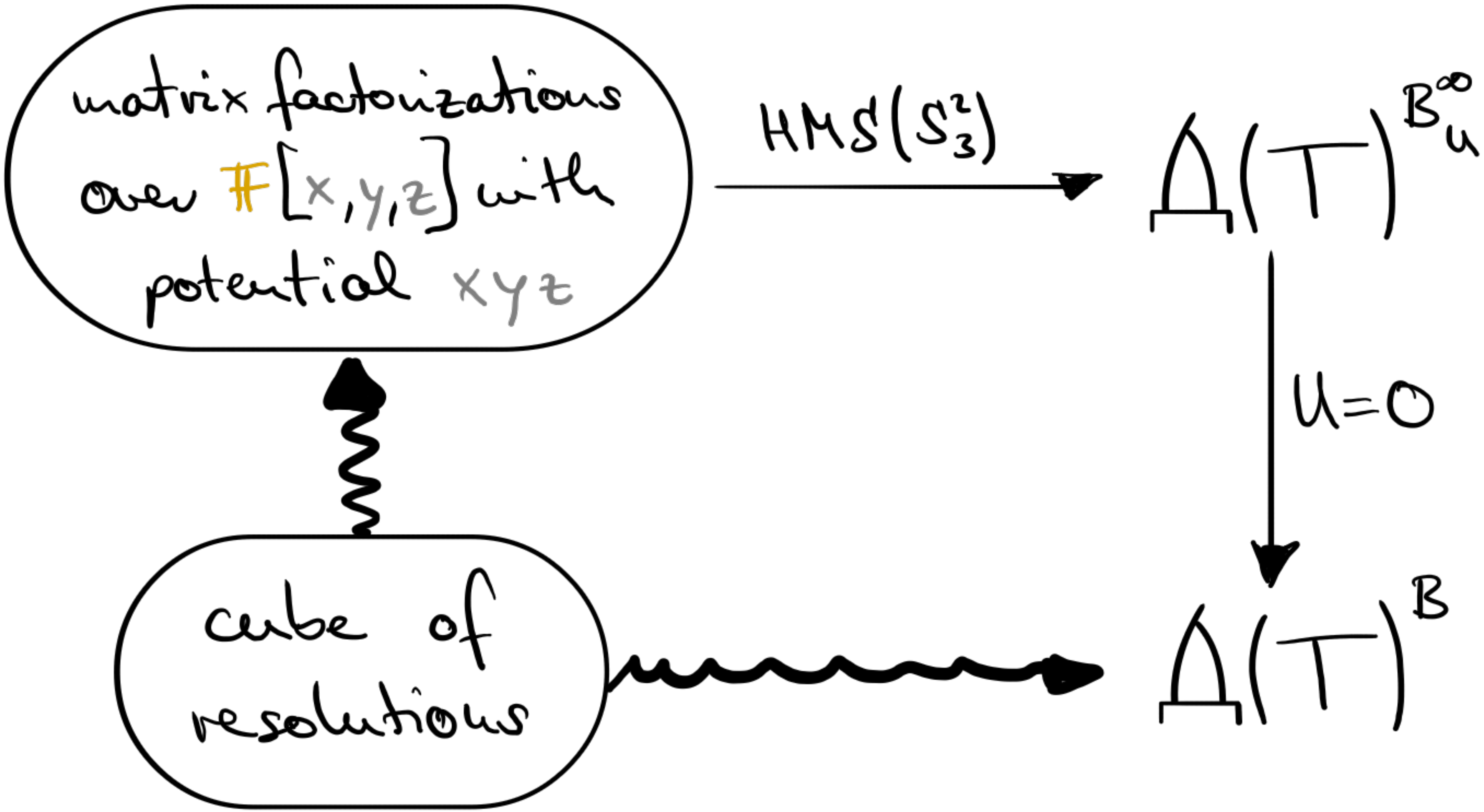
Theorem (Homological mirror symmetry for S^2_3)

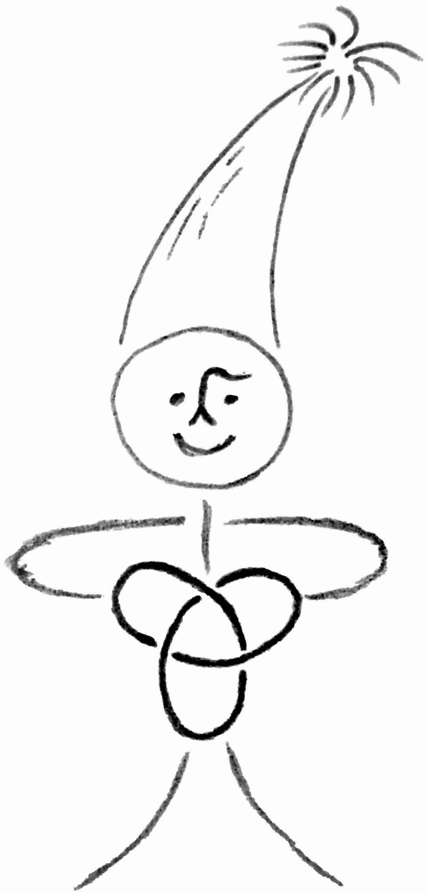
[Abouzaid-Auroux-Efimov-Katzarkov-Orlov'13]

The A_∞-algebra $\mathcal{B}^\infty$ is quasi-isomorphic to the dg algebra

$$\mathcal{A} = \mathrm{End}_{xyz}\left(\begin{array}{c} \mathbb{F}[x,y,z] \\ {\scriptstyle yz}\uparrow\downarrow{\scriptstyle x} \\ \mathbb{F}[x,y,z] \end{array} \oplus \begin{array}{c} \mathbb{F}[x,y,z] \\ {\scriptstyle xz}\uparrow\downarrow{\scriptstyle y} \\ \mathbb{F}[x,y,z] \end{array} \right).$$

Proof strategy for geography theorem





Nagyon köszönöm!


Thank you
for this wonderful event!
(és boldog születésnapot)
Antonio Alfieri
Linh Truong
Irving Dai
Joshua Greene
Andrew Manion
Marco Marengon
Doug Park
András Némethi
Viktória Földvari
Eaman Eftekhary
Vera Vértesi
Sucharit Sarkar
András Juhász
Sarah Rasmussen
Zhongtao Wu