

Mathematik zum Aufassen

Claudius Zibrowius, Professor hier →

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UNIVERSITÄT
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RUB

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Topologie < griech. topos = der Ort >

Mathematik zum Aufpassen

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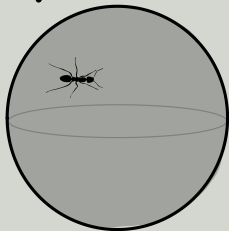
RUB

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Topologie < griech. topos = der Ort >

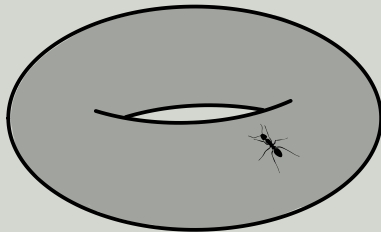
beschäftigt sich mit **Eigenschaften** von
geometrischen Objekten, die unter
stetiger **Verformung** erhalten bleiben.

Sphäre

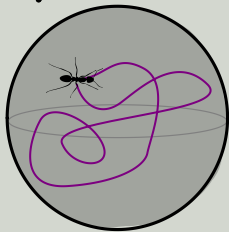


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≠

Torus

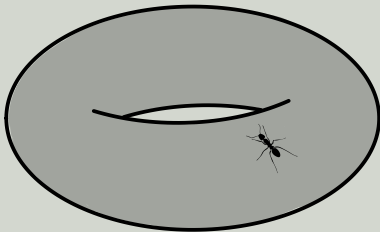


Sphäre

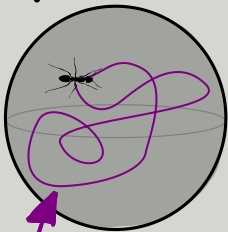


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Torus

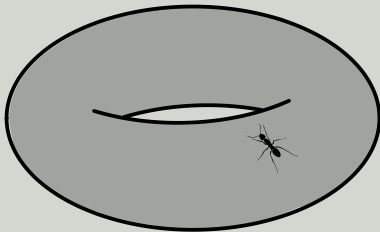


Sphäre



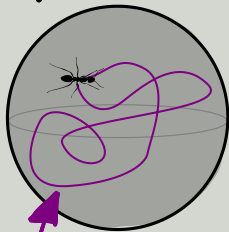
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Torus



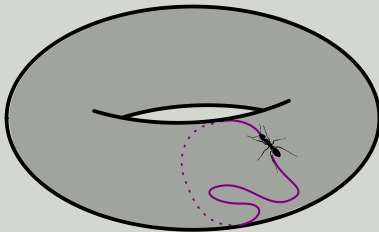
Jede Schleife ist
zusammenziehbar.

Sphäre



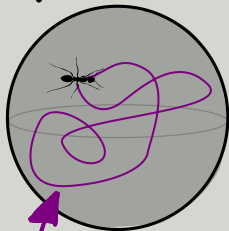
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Torus



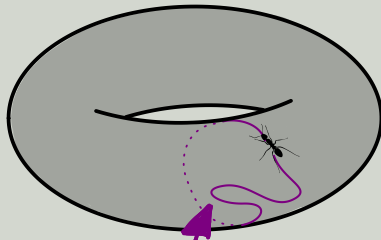
Jede Schleife ist
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Sphäre



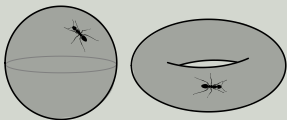
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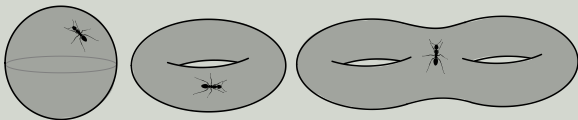
Torus

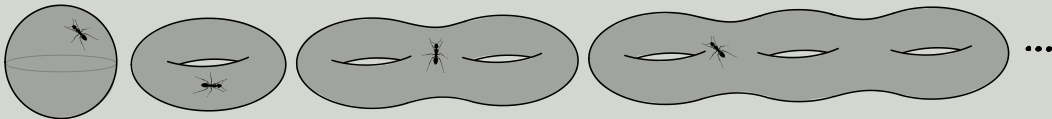


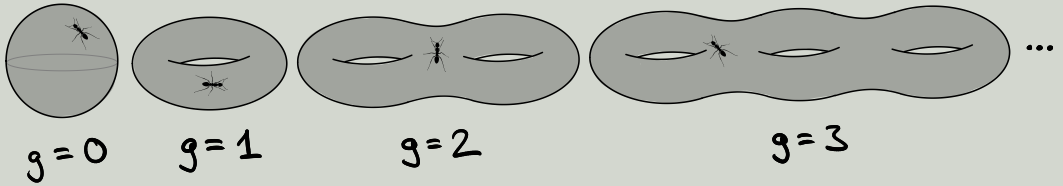
Jede Schleife ist
zusammenziehbar.

Diese Schleife ist
nicht zusammenziehbar.

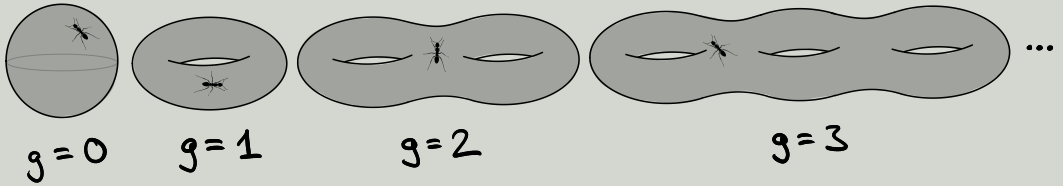




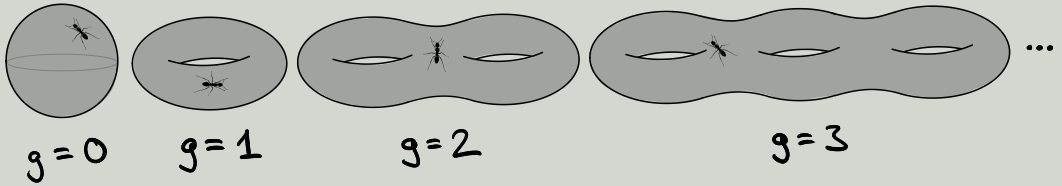




g = Anzahl an "Löchern" in der Fläche

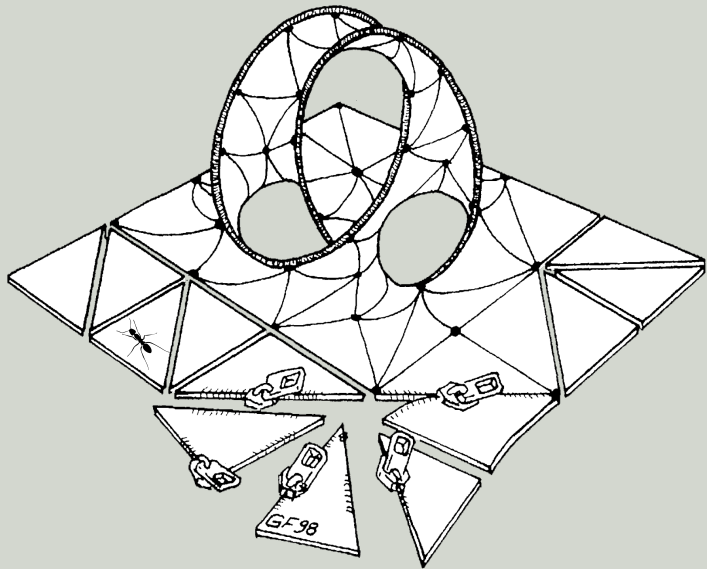


g = Anzahl an "Löchern" in der Fläche
↖ genannt: Geschlecht der Fläche



g = Anzahl an "Löchern" in der Fläche
↖ genannt: Geschlecht der Fläche

Theorem: Jede zweiseitige Fläche ohne Rand lässt sich in eine dieser Standardflächen verformen.



Wie zählt
die Ameise
Löcher?





12 Fünfecke 
+ 20 Sechsecke 

= 32 Flächenstücke



$$\begin{array}{r} 12 \text{ Fünfecke} \\ + 20 \text{ Sechsecke} \end{array}$$

$$= 32 \text{ Flächenstücke}$$

$$\# \text{ Kanten} = \frac{12 \times 5 + 20 \times 6}{2} = 90$$



$$\begin{array}{r} 12 \text{ Fünfecke} \\ + 20 \text{ Sechsecke} \\ \hline \end{array}$$

$$= 32 \text{ Flächenstücke}$$

$$\# \text{ Kanten} = \frac{12 \times 5 + 20 \times 6}{2} = 90$$

$$\# \text{ Ecken} = \frac{12 \times 5 + 20 \times 6}{3} = 60$$



$$\begin{array}{r} 12 \text{ Fünfecke} \\ + 20 \text{ Sechsecke} \\ \hline \end{array}$$

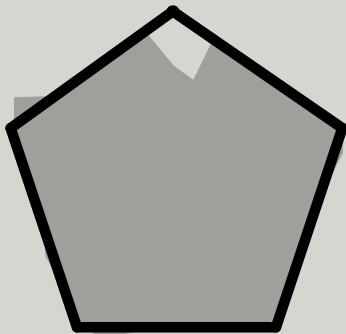
$$= 32 \text{ Flächenstücke}$$

Eulers
Polyederformel

$$\chi = E - K + F$$

$$\# \text{ Kanten} = \frac{12 \times 5 + 20 \times 6}{2} = 90$$

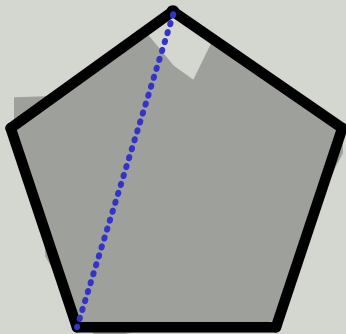
$$\# \text{ Ecken} = \frac{12 \times 5 + 20 \times 6}{3} = 60$$



Eulers

Polyeder formel

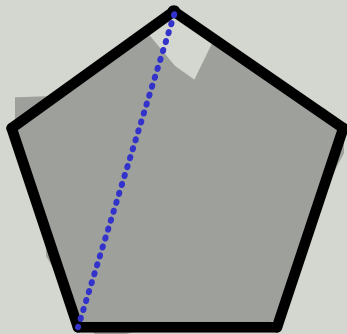
$$\chi = E - K + F$$



Eulers

Polyeder formel

$$\chi = E - K + F$$



Eulers

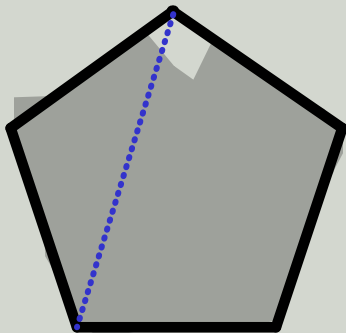
Polyeder formel

$$\chi = E - K + F$$

$$E \mapsto E$$

$$K \mapsto K + 1$$

$$F \mapsto F + 1$$



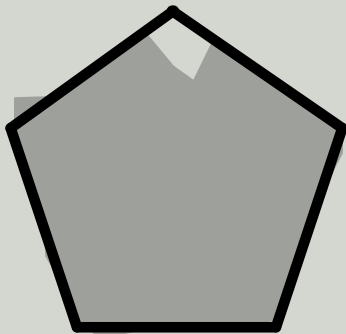
Euler's
Polyeder formula
 $\chi = E - K + F$

$E \mapsto E$

$K \mapsto K + 1$

$F \mapsto F + 1$

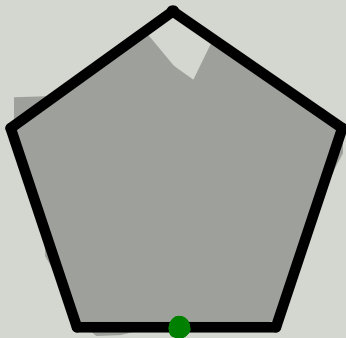
χ konstant



Eulers

Polyeder formel

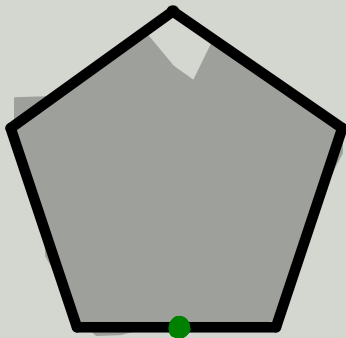
$$\chi = E - K + F$$



Eulers

Polyeder formel

$$\chi = E - K + F$$



Eulers

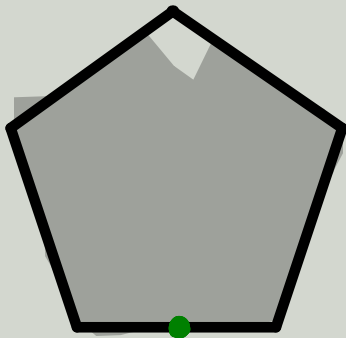
Polyeder formel

$$\chi = E - K + F$$

$$E \mapsto E + 1$$

$$K \mapsto K + 1$$

$$F \mapsto F$$



Eulers

Polyeder formel

$$\chi = E - K + F$$

$$E \mapsto E + 1$$

$$K \mapsto K + 1$$

$$F \mapsto F$$

χ konstant



„Euler-
Charakteristik
von Σ “

Theorem:

Sei Σ eine Fläche.
Dann ist die Zahl

$$\chi(\Sigma) = E - K + F$$

für jede Polygonzerlegung
von Σ konstant.

$$\chi(\text{⚽}) = 2$$

„Euler-
Charakteristik
von Σ “

Theorem:

Sei Σ eine Fläche.

Dann ist die Zahl

$$\chi(\Sigma) = E - K + F$$

für jede Polygonzerlegung
von Σ konstant.

$$\chi(\text{Soccerball}) = 2$$

$$\chi(\text{Lifebuoy}) = 8 - 16 + 8 = 0$$

„Euler-
Charakteristik
von Σ “

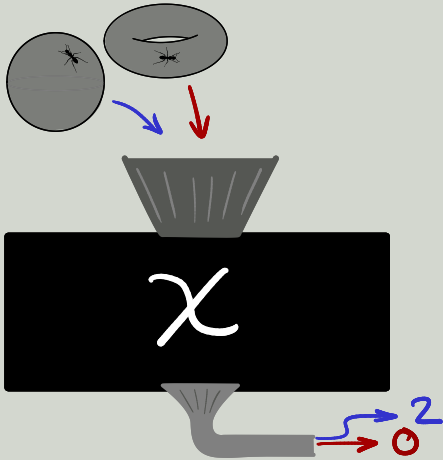
Theorem:

Sei Σ eine Fläche.

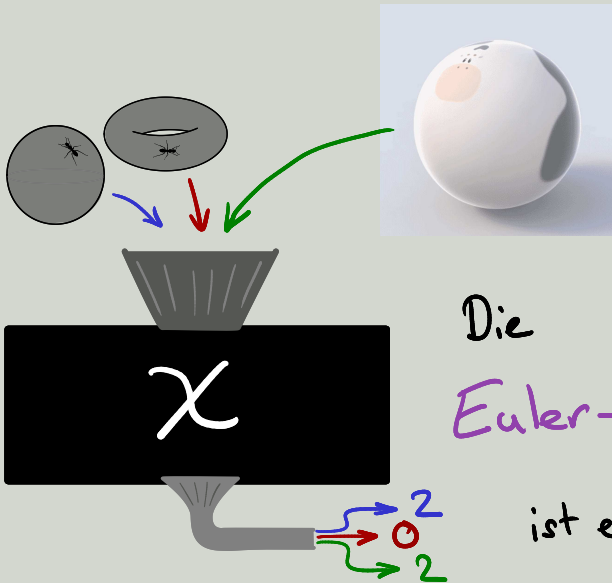
Dann ist die Zahl

$$\chi(\Sigma) = E - K + F$$

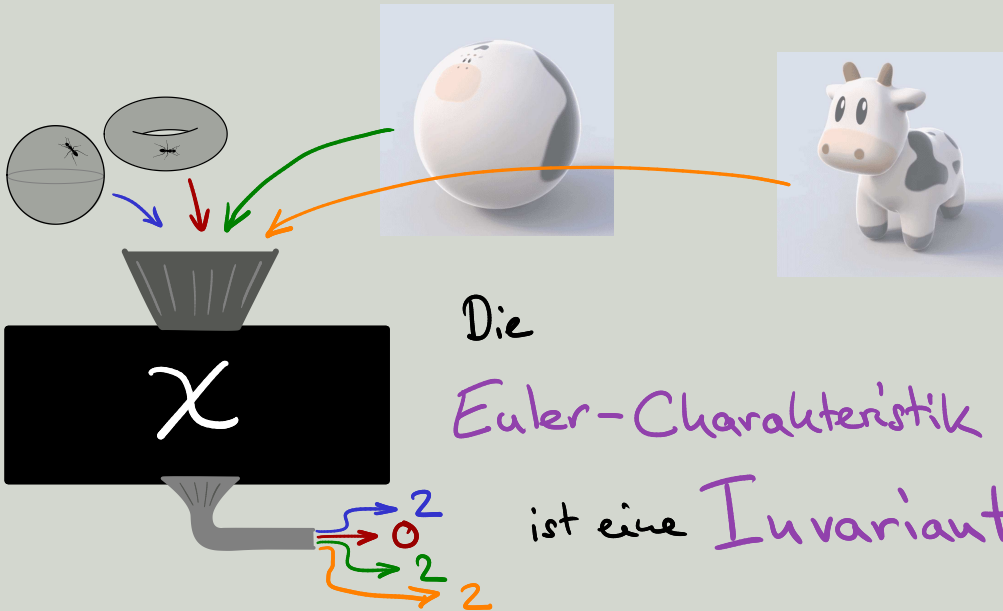
für jede Polygonzerlegung
von Σ konstant.

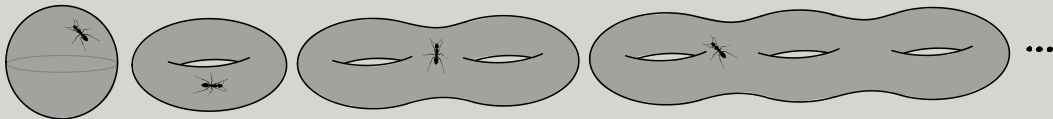


Die
Euler-Charakteristik χ
ist eine Invariante



Die
Euler-Charakteristik χ
ist eine Invariante



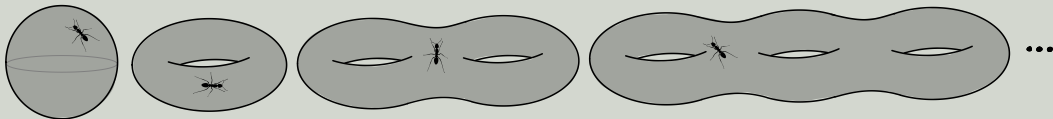


$$\chi=2$$

$$\chi=0$$

$$\chi=-2$$

$$\chi=-4$$



$$\chi = 2$$

$$\chi = 0$$

$$\chi = -2$$

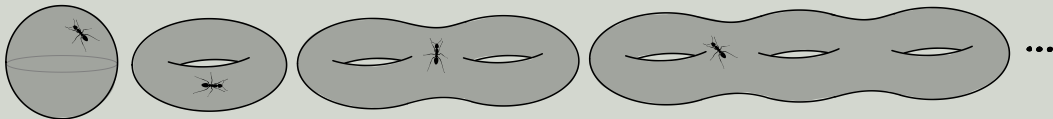
$$\chi = -4$$

$$g = 0$$

$$g = 1$$

$$g = 2$$

$$g = 3$$



$$\chi = 2$$

$$\chi = 0$$

$$\chi = -2$$

$$\chi = -4$$

$$g = 0$$

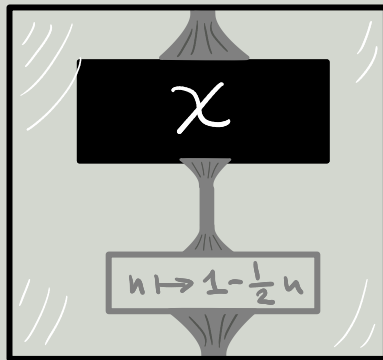
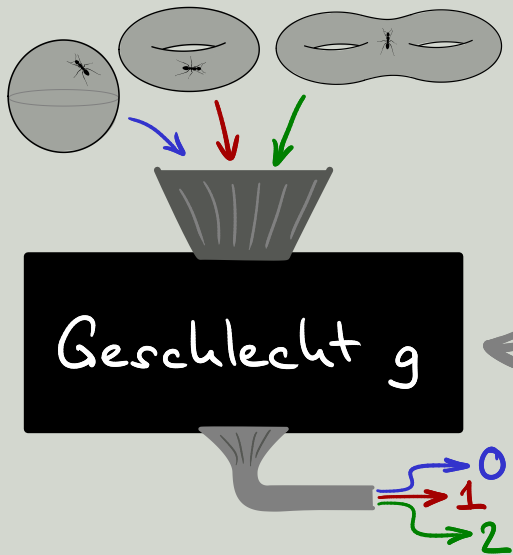
$$g = 1$$

$$g = 2$$

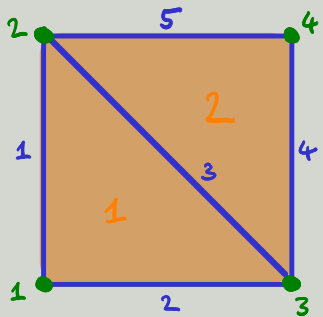
$$g = 3$$

Definition: Geschlecht einer Fläche Σ

$$g(\Sigma) = 1 - \frac{1}{2} \chi(\Sigma)$$



Kategorifizierung



1 ●

2 ●

1 ●

2 ●

3 ●

4 ●

5 ●

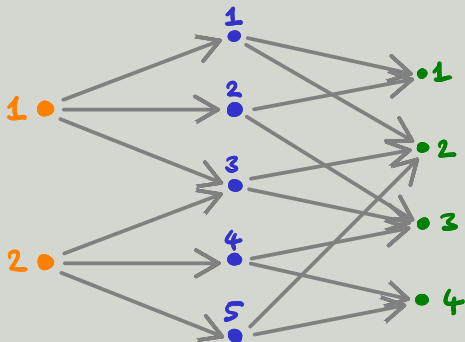
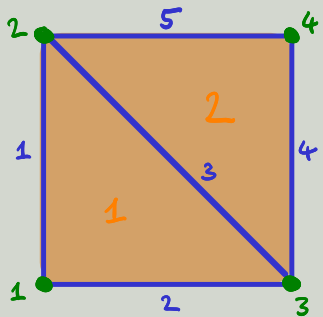
• 1

• 2

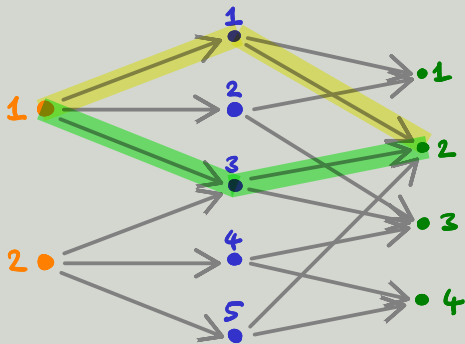
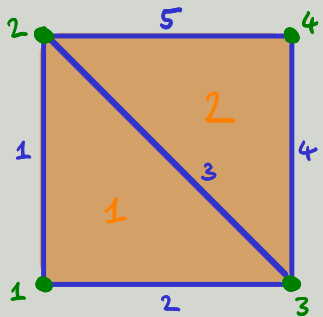
• 3

• 4

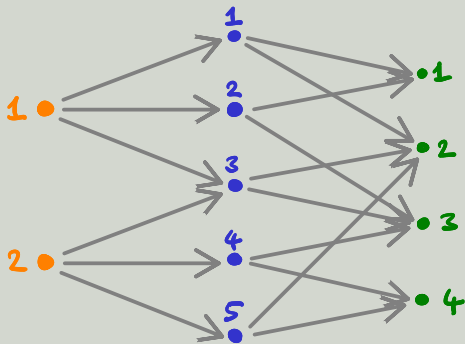
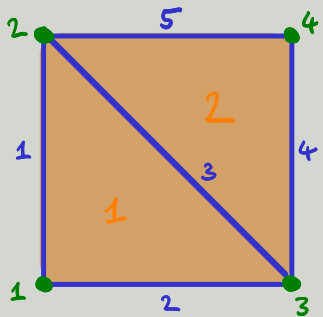
Kategorifizierung



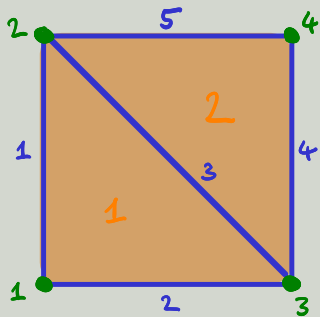
Kategorifizierung



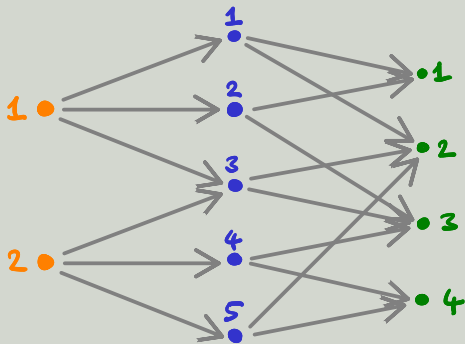
Kategorifizierung



Kategorifizierung

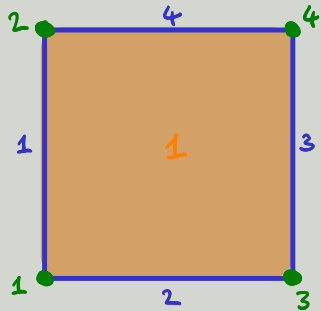


$$d_1 \circ d_2 = 0$$

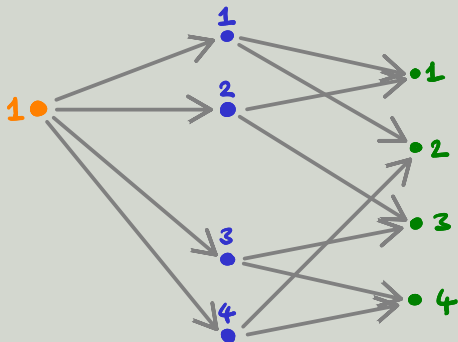


$$\mathbb{R}^2 \xrightarrow{d_2} \mathbb{R}^5 \xrightarrow{d_1} \mathbb{R}^4$$

Kategorifizierung

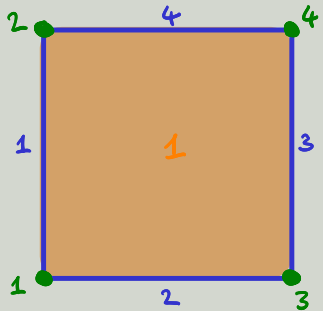


$$d_1 \circ d_2 = 0$$

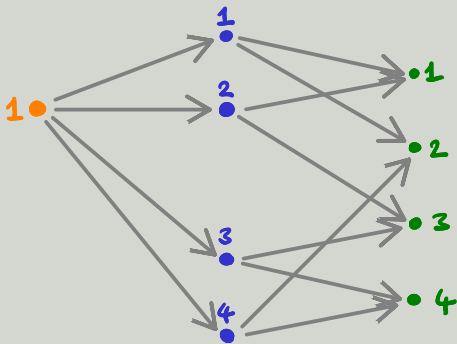


$$\mathbb{R}^1 \xrightarrow{d_2} \mathbb{R}^4 \xrightarrow{d_1} \mathbb{R}^4$$

Kategorifizierung



$$d_1 \circ d_2 = 0$$



$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

Kategorifizierung

$$\begin{aligned}\chi &= E - K + F \\ &= \dim C_0 - \dim C_1 + \dim C_2\end{aligned}$$

$$d_1 \circ d_2 = 0$$

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

Kategorifizierung

$$\begin{aligned}\chi &= E - K + F \\ &= \dim C_0 - \dim C_1 + \dim C_2\end{aligned}$$

$$H_1 = \frac{\ker d_1}{\operatorname{im} d_2}$$

$$d_1 \circ d_2 = 0$$

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

Kategorifizierung

$$\chi = E - K + F$$

$$= \dim C_0 - \dim C_1 + \dim C_2$$

Homologie

$$H_i = \frac{\ker d_i}{\operatorname{im} d_{i+1}}$$

$$d_i \circ d_{i+1} = 0 \quad \dots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} \dots$$

Kategorifizierung

Homologie

$$H_i = \frac{\ker d_i}{\operatorname{im} d_{i+1}}$$

$$\chi = E - K + F$$

$$= \dim C_0 - \dim C_1 + \dim C_2$$

$$= \dim H_0 - \dim H_1 + \dim H_2$$

$$d_i \circ d_{i+1} = 0 \quad \dots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} \dots$$

Kategorifizierung

Emmy Noether
(1882 - 1935)



$$\chi = E - K + F$$

$$= \dim C_0 - \dim C_1 + \dim C_2$$

$$= \dim H_0 - \dim H_1 + \dim H_2$$

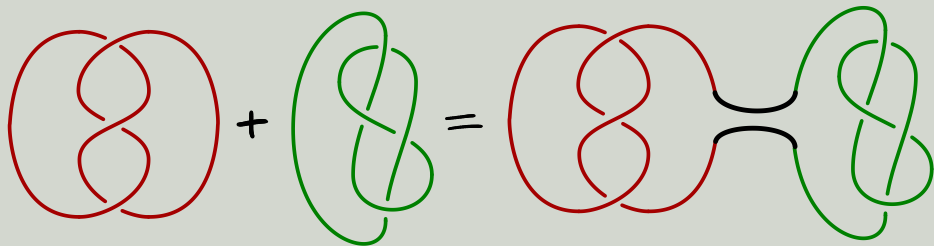
Homologie

$$H_i = \frac{\ker d_i}{\operatorname{im} d_{i+1}}$$

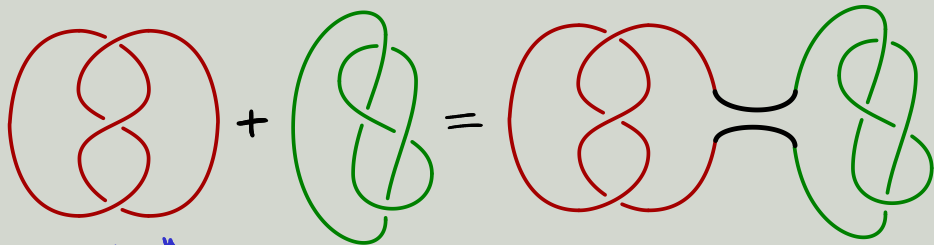
$$d_i \circ d_{i+1} = 0 \quad \dots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} \dots$$

Rechnen mit Knoten

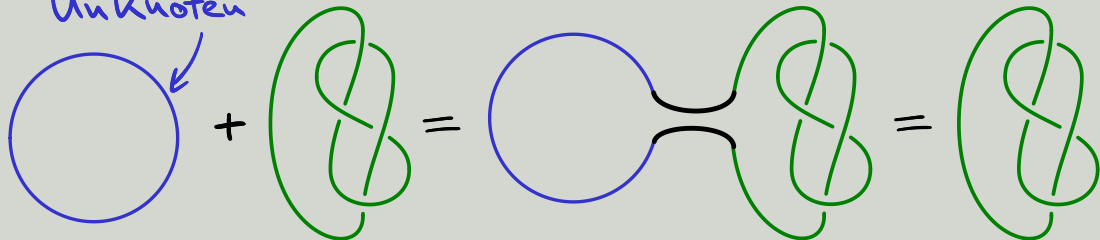
Rechnen mit Knoten



Rechnen mit Knoten



"Unknoten"



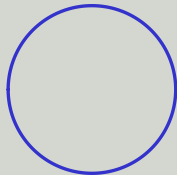
Frage 1:



+

?

=



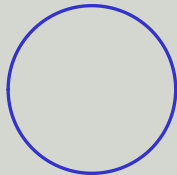
Frage 1:



+

?

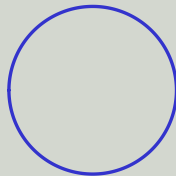
=



Frage 0:



$\overset{?}{\neq}$



~WANTED~

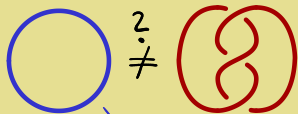


Invariante

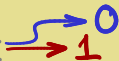


→ 1

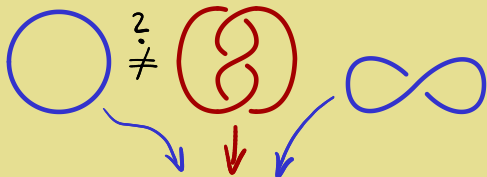
~WANTED~



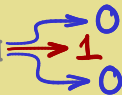
Invariante



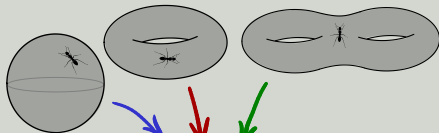
~WANTED~



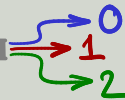
Invariante



~HAVE~



Geschlecht g



Definition: (nach Herbert Seifert, 1897-1996)

Seifertfläche eines Knoten

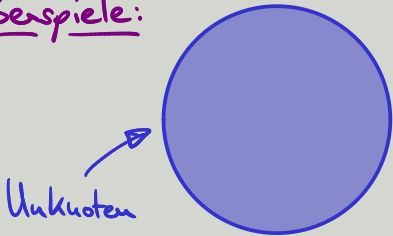
= zweiseitige Fläche, die
vom Knoten berandet wird

Definition: (nach Herbert Seifert, 1897-1996)

Seifertfläche eines Knoten

= zweiseitige Fläche, die
vom Knoten berandet wird

Beispiele:

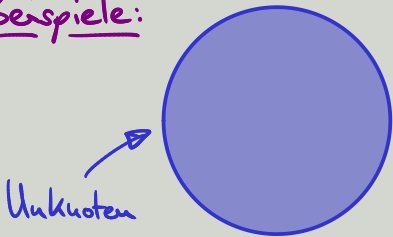


Definition: (nach Herbert Seifert, 1897-1996)

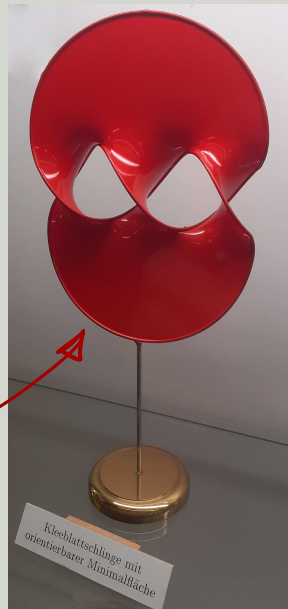
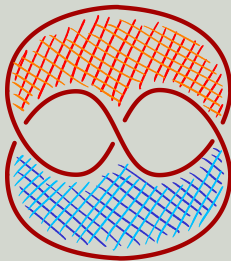
Seifertfläche eines Knoten

= zweiseitige Fläche, die
vom Knoten berandet wird

Beispiele:



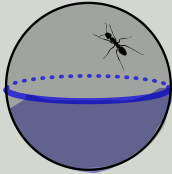
Unknoten



Was ist das Geschlecht einer Seifertfläche?

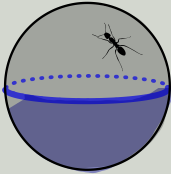


Was ist das Geschlecht einer Seifertfläche?

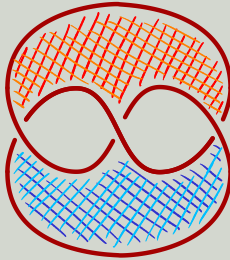


$$\Rightarrow g = 0$$

Was ist das Geschlecht einer Seifertfläche?

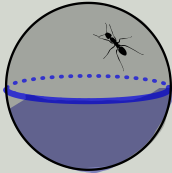


$$\Rightarrow g = 0$$

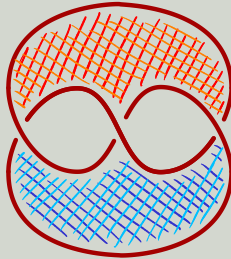


Kleeblattschlinge mit
orientierbarer Minimalfläche

Was ist das Geschlecht einer Seifertfläche?



$$\Rightarrow g = 0$$

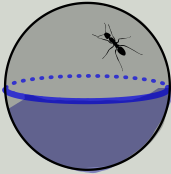


$$\chi = E - K + F$$

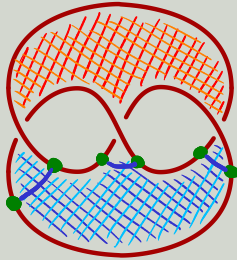


Kleeblattschlinge mit
orientierbarer Minimalfläche

Was ist das Geschlecht einer Seifertfläche?



$$\Rightarrow g = 0$$

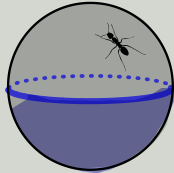


$$\chi = E - K + F$$

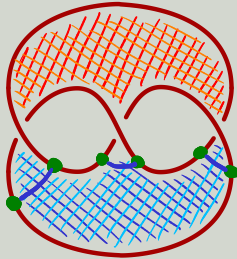


Kleeblattschlinge mit
orientierbarer Minimalfläche

Was ist das Geschlecht einer Seifertfläche?



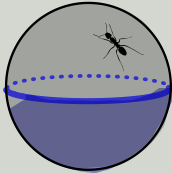
$$\Rightarrow g = 0$$



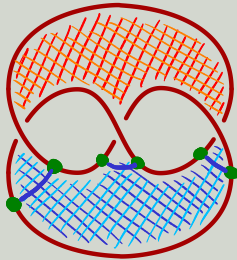
$$\begin{aligned}\chi &= E - K + F \\ &= 6 - 9 + 2 = -1\end{aligned}$$



Was ist das Geschlecht einer Seifertfläche?

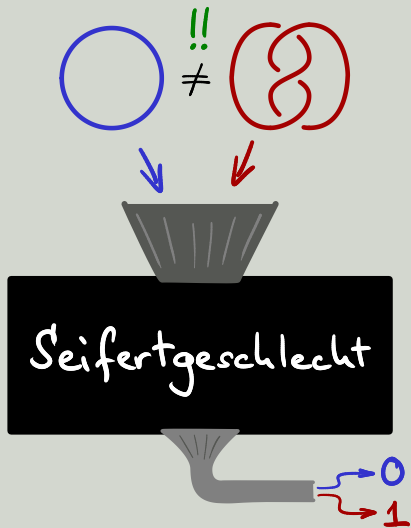


$$\Rightarrow g = 0$$



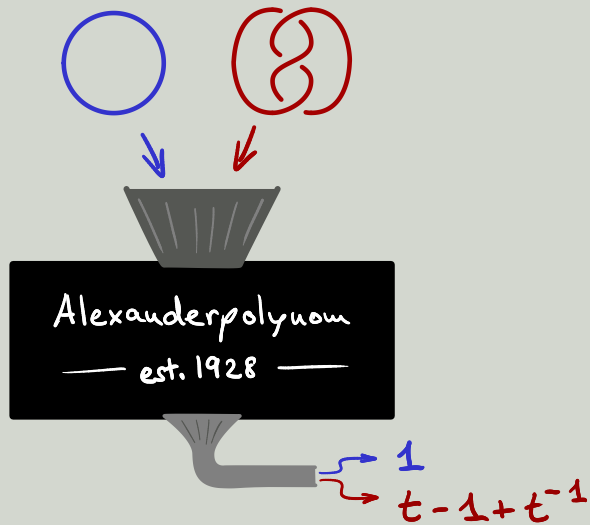
$$\begin{aligned}\chi &= E - K + F \\ &= 6 - 9 + 2 = -1 \\ &\Rightarrow g = 1\end{aligned}$$

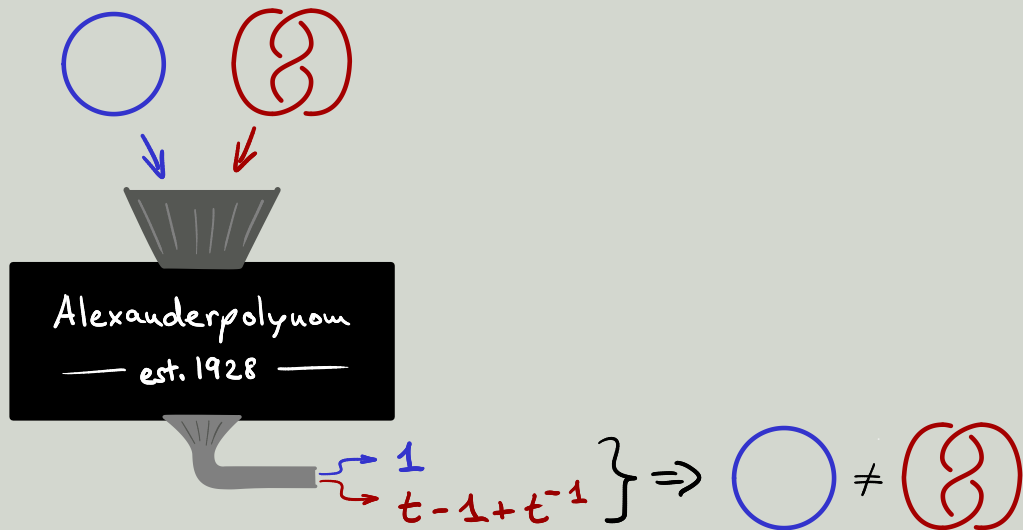


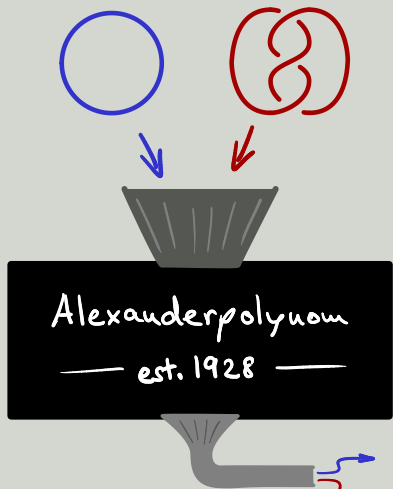


Definition
 Seifertgeschlecht
 eines Knoten
 =
 minimale Geschlecht
 von Seifertflächen
 für den Knoten







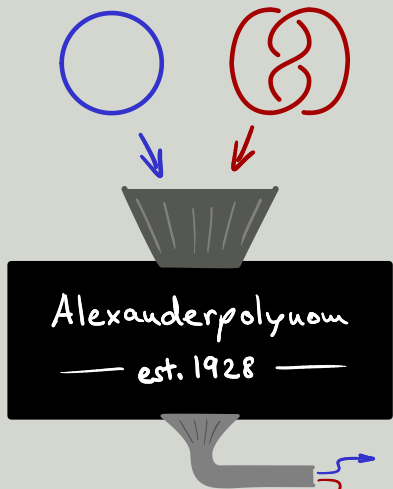


Kategorifizierung

(Ozsváth-Szabó, Rasmussen 2001)

Knoten-Floer-Homologie

$$\{1, t-1+t^{-1}\} \Rightarrow \bigcirc \neq \text{figure-eight knot}$$



Alexanderpolynom
— est. 1928 —

$\begin{matrix} \rightarrow 1 \\ \rightarrow t-1+t^{-1} \end{matrix}$

$\}$

$\Rightarrow$



$\neq$



Kategorifizierung

(Ozsváth-Szabó, Rasmussen 2001)

Knoten-Floer-Homologie



Andreas Floer (1956-1991, RUB)

Gleichung $\textcircled{8} + X = \textcircled{}$

Theorem: Die Gleichung $\text{⌘} + \text{X} = \text{◯}$
hat keine Lösung.

Theorem: Die Gleichung $\textcircled{8} + X = \textcircled{}$
hat keine Lösung.

Warum?

Theorem: Die Gleichung $(\infty) + X = \bigcirc$
hat keine Lösung.

Warum?

$$g((\infty) + X) = g(\bigcirc) = 0$$

Theorem: Die Gleichung $(\infty) + X = \bigcirc$
hat keine Lösung.

Warum?

$$g(\infty) + g(X) = g(\infty + X) = g(\bigcirc) = 0$$

Theorem: Die Gleichung $(\infty) + X = \bigcirc$
hat keine Lösung.

Warum?

$$\underbrace{g(\infty)}_{=1} + \underbrace{g(X)}_{\geq 0} = g(\infty + X) = g(\bigcirc) = 0$$

Test

Nehmen Sie ein unverknotetes Seil mit beiden Händen an den Seilenden auf und verknoten Sie das Seil, ohne es dabei loszulassen.

