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L-space knots have no essential Conway spheres

joint work with Tye Lidman and Allison Moore

arXiv: 2006.03521

L-spaces

Let M be some n-manifold for some integer n. Then

$$\operatorname{rk} \mathrm{H}_*(M) \geq |\chi(\mathrm{H}_*(M))|.$$

Question

Can we characterize those M for which we have "$=$"?

Similarly, let Y be some 3-manifold and let $\widehat{\mathrm{HF}}(Y)$ denote the Heegaard Floer homology of Y. Then

$$\operatorname{rk} \widehat{\mathrm{HF}}(Y) \geq |\chi(\widehat{\mathrm{HF}}(Y))|.$$

Definition

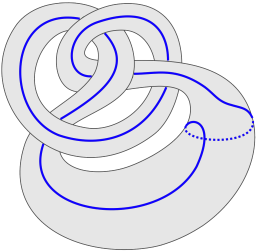
Y is called an **L-space** if we have "$=$".

known facts:

- Y is an L-space $\Leftrightarrow \begin{cases} Y \text{ is a } \mathbb{Q}HS^3 \text{ and} \\ \operatorname{rk} \widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|. \end{cases}$
- Ozsváth–Szabó: $\{\text{L-space}\} \supset \{Y \mid \pi_1(Y) \text{ finite}\} \supset \{\text{lens space}\}$
- Ozsváth–Szabó: L-spaces do not admit cooriented taut foliations.
- Boyer–Gordon–Watson–Juhász: L-space conjecture (predicts two Heegaard Floer independent characterizations of L-spaces)

L-space knots

Recall surgery along a knot $K \subset S^3$:



$S^3_{p/q}(K)$

Let $X_K := S^3 \smallsetminus \mathrm{nbhd}(K)$.

Given $p/q \in \mathbb{Q}P^1$, define

$$S^3_{p/q}(K) := X_K \cup_h D^2 \times S^1,$$

where

$$h\colon \partial(D^2 \times S^1) \xrightarrow{\simeq} \partial X_K$$

and $h|_{\partial D^2 \times \{*\}}$ is a curve γ such that

$$[\gamma] = p\mu_K + q\lambda_K \in H_1(\partial X_K).$$

observations:

- trivial surgery: $S^3_\infty(K) = S^3$ is an L-space.
- 0-surgery: $S^3_0(K)$ is not an L-space ($b_1 > 0$).

Definition

K is called an **L-space knot** if there is some $\frac{p}{q} \in \mathbb{Q}^{>0}$ such that $S^3_{p/q}(K)$ is an L-space.

known facts:

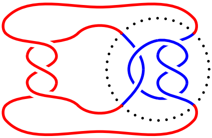
- For any knot K, the set of L-space surgery slopes is either $\{\infty\}$ or an interval (Rasmussen[2]). If it is an interval, either K or its mirror K^* is an L-space knot.
- L-space knots are:
  - fibred (Ghiggini, Ni)
  - prime (Krcatovich, Hedden–Watson, Baldwin–Vela-Vick)

Conway spheres

Definition (named after John Conway, 1937–2020)

Given a knot $K \subset S^3$, an embedded sphere $S^2 \subset S^3$ is a **Conway sphere** for K if it intersects K transversely and in precisely four points.

For example:

$T(3,-4) =$ 

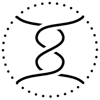
A Conway sphere splits a knot into two 2-string tangles:

$$K = T_1 \cup T_2$$

Definition

A 2-string tangle is called **split** if the two strands can be separated by an embedded disk.

split



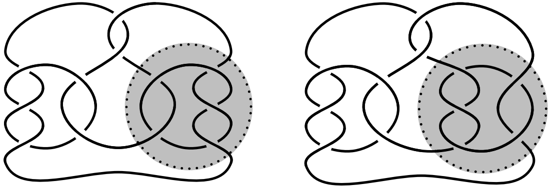
split

non-split

Definition

A Conway sphere is **essential** if neither T_1 nor T_2 is split.

Conway mutation



Kinoshita-Terasaka knot   Conway knot

known fact: Conway mutation on inessential Conway spheres preserves knots.

Main results

Theorem (Lidman–Moore–Z'20)

No L-space knot admits an essential Conway sphere.

This had been conjectured by Lidman and Moore in 2013.

Corollary (Lidman–Moore–Z'20)

Conway mutation preserves L-space knots.

Corollary (Wu'96, Lidman–Moore–Z'20)

Let K be a knot in S^3 with an essential Conway sphere. Then $\pi_1(S_{p/q}(K))$ is infinite for all $p/q \in \mathbb{Q}$.

A structure theorem for $\widehat{\mathrm{HFK}}$ of L-space knots

Knot Floer homology: (Ozsváth–Szabó, Rasmussen)

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{A,M\in\mathbb{Z}} \widehat{\mathrm{HFK}}_M(K;A)$$

where

A is the Alexander grading and
M is the Maslov grading.

Theorem (Ozsváth–Szabó'03)

If a knot K or its mirror is an L-space knot, then

$$\widehat{\mathrm{HFK}}(K) \cong \bigoplus_{k=-\ell}^{\ell} \mathbb{F}_{(M_k,A_k)},$$

where

$$M_k < M_{k+1} \quad \text{and} \quad A_k < A_{k+1} \quad \text{for all } k.$$

Further, if K is an L-space knot, then

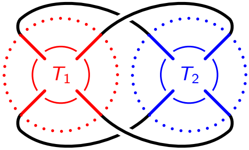
$$M_k = M_{k-1} + 1 \quad \text{for all } k \equiv \ell + 1 \mod 2.$$

Lemma (known to experts, Lidman–Moore–Z'20)

The converse of the first part also holds.

Computing $\widehat{\mathrm{HFK}}$ from 2-string tangle decompositions

convention:

$K = T_1 \cup T_2 =$ 

strategy: find a tangle invariant $\mathrm{HFT}(T)$ that

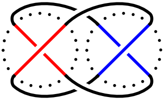
a) detects split tangles

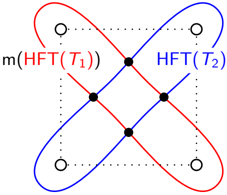
b) satisfies a gluing theorem of the form

$$\widehat{\mathrm{HFK}}(K) \otimes \mathbb{F}^i \cong \mathrm{HF}(\mathrm{m}(\mathrm{HFT}(T_1)), \mathrm{HFT}(T_2)),$$

where HF = Lagrangian Floer homology and $i \in \{1, 2\}$, eg:

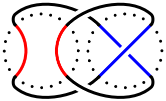
$K =$ Hopf link

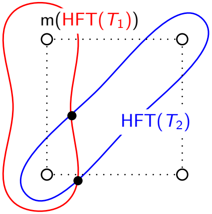




$\widehat{\mathrm{HFK}}(K) = \mathbb{F}^4$

$K =$ unknot





$\widehat{\mathrm{HFK}}(K) = \mathbb{F}^1$

The tangle invariant HFT

Theorem (Z'17)

There exists a map

$$\left.\left\{\begin{array}{c}\text{2-string tangle } T \\ \text{in a 3-dim. ball } B^3\end{array}\right\}\right/\text{isotopy} \to \left.\left\{\begin{array}{c}\text{immersed curves* } \mathrm{HFT}(T) \\ \text{on } \partial B^3 \smallsetminus \partial T\end{array}\right\}\right/\text{homotopy}$$

*) plus local systems $X \in \mathrm{GL}_n(\mathbb{F})$

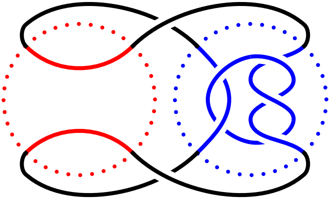
such that

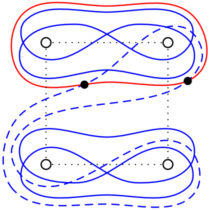
$$\widehat{\mathrm{HFK}}(K) \otimes \mathbb{F}^2 \cong \mathrm{HF}(\mathrm{m}(\mathrm{HFT}(T_1)), \mathrm{HFT}(T_2)),$$

for any decomposition $K = T_1 \cup T_2$ of a knot (and similar for links).

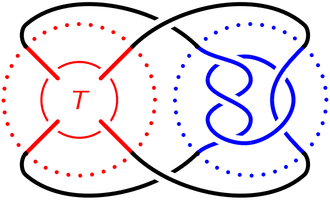
- The proof relies on Zarev's bordered sutured HF theory.
- In this talk, I will treat the construction of $\mathrm{HFT}(T)$ as a black box ■.
- $\mathrm{HFT}(T)$ can be equipped with a bigrading which is compatible with gluing.

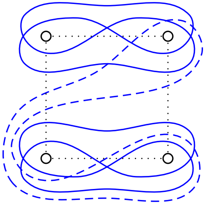
example:





$\widehat{\mathrm{HFK}}$ and Conway mutation

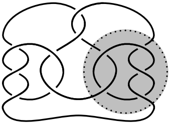




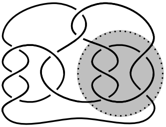
Theorem (mutation symmetry, Z'19)

For any 2-string tangle T, $\mathrm{HFT}(\circlearrowleft_\pi(T)) = \mathrm{HFT}(T)$.

There exist refinements of this theorem with gradings.



Kinoshita-Terasaka knot



Conway knot

Corollary (mutation conjecture for $\widehat{\mathrm{HFK}}$, Z'19)

Conway mutation preserves relatively δ-graded $\widehat{\mathrm{HFK}}(K)$ *for any link K.*

This had been conjectured by Baldwin and Levine in 2011.

Symmetries for HFT

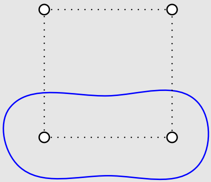
The proof of mutation symmetry for HFT is based on two intermediate results:

- geography of components of HFT
- action of conjugation bimodule

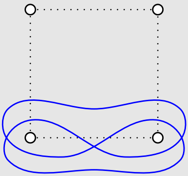
Theorem (geography, Z'19)

Up to reparametrization, every component of HFT(T) *is either*

rational *or* ***special****:*



(embedded curve separating two pairs of punctures)



(immersed curve wrapping around one pair of punctures at least once)

Observation (HFT detects rational tangles, Z'17)

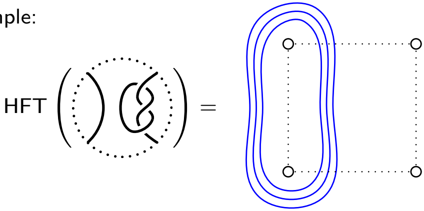
A 2-string tangle is rational, ie isotopic to *, iff* HFT(T) *consists of a single rational component (with 1-dim. trivial local system).*

Theorem (conjugation symmetry, Z'19)

Special components come in conjugate pairs.

HFT detects split tangles

example:



Theorem (Lidman–Moore–Z'20)

A 2-string tangle T is split iff HFT(T) *consists of parallel rational components only.*

The proof is based on Juhász's sutured HF theory.

Theorem (Lidman–Moore–Z'20)

For any 2-string tangle T without closed components, the number of rational components in HFT(T) *is odd.*

Proof of main result.

- Fix some Conway sphere decomposition of some knot

$$K = T_1 \cup T_2.$$

- Suppose neither T_1 nor T_2 is split.
- Then, HFT(T_1) and HFT(T_2) are "sufficiently complicated" such that

$$\widehat{\mathrm{HFK}}(K) \neq \widehat{\mathrm{HFK}}(\text{L-space knot}). \qquad \square$$

Open questions

Definition

A knot K is n-string prime if any sphere intersecting K transversely and in $2n$ points defines an essential surface in the knot exterior.

- 1-string prime = prime
- 2-string prime = no essential Conway sphere

Conjecture (Baker–Moore'14)

L-space knots are n-string prime for any positive integer n.

Theorem (Baker–Motegi'19)

The conjecture is true for satellite knots if $n \leq 3$.

Motegi pointed out the following corollary of our main result:

Corollary

The conjecture is true for satellite knots if $n \leq 5$.

The proof of our main result raises the following questions:

Questions

Suppose K is a knot whose $\widehat{\mathrm{HFK}}$ is at most 1-dimensional in each Alexander grading and that K is not an L-space knot.

- *Does K admit an essential Conway sphere?*
- *In fact, is there any such knot?*

Thank  you!