Khovanov Stable Homotopy

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Part III Essay

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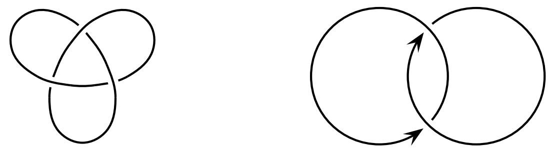
0 Introduction

Most ideas presented in this essay are fairly recent in origin: Our starting point is the Jones polynomial, an invariant of knots and (more generally) oriented links in S^3. This invariant has only been known since the early 1980s, although it is surprisingly easy to calculate. One starts with a diagram representing the given knot or oriented link, i. e. a generic planar projection of it. Examples of such diagrams are the following visualisations of the so-called left-handed trefoil knot (*left*) and the Hopf link with some orientation indicated by the arrows (*right*):



Given such a diagram, the Jones polynomial can be computed using an algorithm called the Kauffman state model, which will be the first thing to discuss in Section 1.
In 1999, Khovanov generalised this construction [K99]. To a given diagram, Khovanov assigns a graded chain complex, i. e. a chain complex with two gradings: In addition to the "normal" homological grading, the chain groups also carry a second grading, called the "quantum grading". This chain complex – the Khovanov chain complex – is constructed in such a way that it returns the Jones polynomial as its graded Euler characteristic. The chain complex itself is not a link invariant; however, different diagrams of the same oriented link are chain homotopy equivalent, so their homology – the Khovanov homology – is indeed an invariant. In Section 1, we discuss the definition of the Khovanov chain complex, introducing the notation necessary for subsequent sections.
The main part of this essay, Section 2, is concerned with one particular generalisation of Khovanov homology, which was first described by Lipshitz and Sarkar in 2011 [LS11]. To an oriented link diagram, they assign a CW complex whose cellular chain complex is the Khovanov chain complex (up to some degree shift, which depends on some choices in the construction). The extra grading is achieved by first constructing a CW complex for each grading separately and then taking the wedge product of these spaces, but "remembering" the grading. By construction, the relationship between this "Khovanov homotopy type" and Khovanov homology is very simple: The cohomology of the CW complexes returns the Khovanov homology groups (again, up to some degree shift). Unfortunately, the CW complexes themselves are not link invariants; however, all CW complexes constructed for the same oriented link are stably homotopy equivalent.
In Section 3, we briefly discuss Lipshitz and Sarkar's explicit formulas for computing the first two Steenrod squares on Khovanov homology in terms of the generators of the Khovanov chain groups [LS12]. These computations are motivated by the question whether the constructed Khovanov homotopy type actually contains more information about an oriented link than Khovanov homology. As stable cohomology operations, the Steenrod squares offer a means of distinguishing spaces which are not stably homotopy equivalent. Based on Lipshitz and Sarkar's computations, Seed found an example of two link diagrams with equal Khovanov homology groups, but different Khovanov homotopy types [S12], thus giving a positive answer to the question above.

1 Khovanov Homology

1.1 From the Jones Polynomial to Khovanov Homology

In this subsection, we contrast the definition of the Jones polynomial with the (very similar) construction of Khovanov homology. We follow the discussion in Bar-Natan's paper [BN02] using his notation, before introducing Lipshitz and Sarkar's slightly different notation in Section 1.2, which is more appropriate for our purposes.

The Jones Polynomial. Let L be an link diagram on S^2. At each crossing, we can alter the diagram in essentially two ways such that the crossing is "resolved", namely by ⤫ → ≍ and ⤫ →)(. These two "states" are called 0- and 1-resolutions respectively. We define the Jones polynomial in terms of the Kauffman bracket $\langle\cdot\rangle$, which turns a given link diagram into a polynomial in one variable q. It is defined recursively by the formulas

$$\langle\emptyset\rangle = 1, \quad \langle L\bigcirc\rangle = (q+q^{-1})\langle L\rangle \quad \text{and} \quad \langle ⤫ \rangle = \langle ≍ \rangle - q\langle)(\rangle. \tag{1}$$

Then, for an oriented link, the unnormalised Jones polynomial is defined as

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-}\langle L\rangle,$$

where n_+ is the number of positive crossings ⤫ and n_- the number of negative ones ⤫. Note that the Jones polynomial of a knot does not depend on its orientation. $J(L)$ is an invariant of the oriented link corresponding to the diagram L, which is not hard to show, using the three Reidemeister moves. There is also a normalised version of the Jones polynomial, which is obtained by dividing $J(L)$ by $(q+q^{-1})$, so that the Jones polynomial of the unknot becomes 1.
It is useful to adopt another point of view on the Kauffman bracket. Instead of resolving one crossing at a time, we can resolve all crossings at once to get a complete resolution of a given link diagram L. If there are n crossings in total, each of these complete resolutions corresponds bijectively to a vertex in $\{0,1\}^n$, since we have two choices – 0-resolution or 1-resolution – at each crossing. We can then give an explicit formula for the Kauffman bracket: For every circle in a complete resolution, we get a factor $(q+q^{-1})$ and each 1-resolution contributes a factor $(-q)$, so

$$\langle L\rangle = \sum_{v\in\{0,1\}^n} (q+q^{-1})^{c(v)}(-q)^{|v|}, \tag{2}$$

where $c(v)$ is the number of circles in the resolution corresponding to the vertex v and $|v|$ is the sum over all entries of v, i. e. the number of 1-resolutions.

Khovanov Homology. Khovanov's idea is to substitute the Kauffman bracket by a chain complex of graded modules such that each summand in (2) can be interpreted as the graded dimension of a graded module associated to each resolution. Let V be the graded $\mathbb{Z}$-module with the two generators x_- and x_+ in grading -1 and $+1$ respectively. For each $v\in\{0,1\}^n$, we define $V(v) := V^{\otimes c(v)}\{|v|\}$, where $\cdot\{i\}$ denotes a shift in gradings: elements of degree j become elements of degree $j+i$. Then the graded dimension $\dim_q(V(v))$ of $V(v)$ equals $(q+q^{-1})^{c(v)}q^{|v|}$. If we compare this to the summands in (2), it only differs by the factor $(-1)^{|v|}$. So if we define the module of homological grading i

in the chain complex to be the direct sum of those $V(v)$ with $|v| = i$, the graded Euler characteristic of the chain complex is the Kauffman bracket of the corresponding link diagram, provided the boundary maps preserve degrees. In fact, it is possible to define such boundary maps, which we do in Section 1.3, when we have more notation.
As mentioned above, we can identify the set of resolutions of a given link diagram L with $\{0,1\}^n$, the set of vertices of the standard n-dimensional cube. It turns out to be a good idea to suggest that the boundary maps should correspond to the edges of this cube. The endpoints of these edges differ in just one entry. By convention, the gradings increase along the boundaries (so the Khovanov chain complex is in fact a cochain complex). Hence each boundary map should correspond to turning a 0-resolution into a 1-resolution.
Suppose we have such boundary maps. Then we can give a recursive description of the above which corresponds to (1). We define the Khovanov bracket $[\![\cdot]\!]$ by

$$[\![\emptyset]\!] = 0 \to \mathbb{Z} \to 0, \quad [\![L\bigcirc]\!] = [\![L]\!] \otimes V \quad \text{and} \quad [\![\times]\!] = \mathcal{F}\left(0 \to [\![\asymp]\!] \xrightarrow{d} [\![)\,(]\!]\{1\} \to 0\right).$$

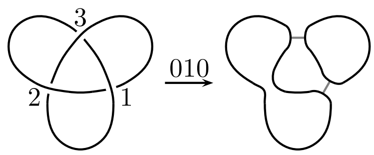
Here, $\mathcal{F}$ denotes the "flatten" operator which turns the double complex into an ordinary chain complex by taking direct sums over those modules of same homological gradings. (Bar-Natan suggests this notation in [BN02].) d denotes the boundary maps.
Since the Jones polynomial is obtained by multiplying the Kauffman bracket by a certain factor which depends on the orientation of the link diagram, we also have to take this into account for the Khovanov chain complex. The effect of the factor $q^{n_+ - 2n_-}$ can be mimicked by increasing the quantum grading of the chain module by $n_+ - 2n_-$. The factor $(-1)^{n_-}$ suggests a shift in homological gradings, which we denote by $\cdot[i]$ in analogy to shifts $\cdot\{i\}$ in quantum gradings. It turns out that the homology (or rather cohomology) of $[\![L]\!]\{n_+ - 2n_-\}[n_-]$ is indeed a link invariant. By construction, its Euler characteristic returns the Jones polynomial.

1.2 Resolution Configurations

In this section, we introduce some notation that will be useful for the construction of the boundary maps of the Khovanov chain complex. Even more importantly, it is fundamental for Sections 2 and 3.

Definition. A **resolution configuration** D consists of the following data: A finite set $Z(D)$ of pairwise disjoint circles embedded in S^2, a finite set $A(D)$ of pairwise disjoint arcs in S^2 such that $A(D) \cap Z(D) = \partial A(D)$, and a total order on $A(D)$. If $Z \cap A \neq \emptyset$ for some $Z \in Z(D)$ and $A \in A(D)$, we say that Z is an **endpoint** of A. $|A(D)|$ is called the **index** $\mathrm{ind}(D)$ of D.

This definition is, of course, motivated by the resolutions which we considered in the previous section. Given a link diagram L with, say, n crossings, choose a total order on the set of crossings, which gives rise to a bijection between $\{0,1\}^n$ and the set of complete resolutions. Then each such complete resolution can be considered as a resolution configuration, denoted by $D_L(v)$ where v is the corresponding vector in $\{0,1\}^n$. At every 0-resolution, we add an arc like this $\times \to \asymp$ and the total ordering on $A(D_L(v))$ is induced by the one on the crossings of L. An example of this is shown on the next page.



The purpose of the arcs is to "remember" the 0-resolutions, as the boundary maps are defined by changing a 0-resolution into a 1-resolution: $\asymp \to)($. Such an altering of the diagram can also be described by **embedded surgery** along the arc: If we delete two disks ($\cong S^0 \times D^1$) around the endpoints of the arc, we get four distinct points $\partial(S^0 \times D^1) = S^0 \times S^0$. We then write $S^0 \times S^0 = \partial(D^1 \times S^0)$, so we get can glue $D^1 \times S^0$ in and get $\asymp \to \vdots\,\vdots \to)($.

Definition. Let $A' \subseteq A(D)$ for some resolution configuration D. We define $s_{A'}(D)$ to be the resolution configuration obtained by perfoming embedded surgery along the arcs in A'. The total order on the remaining arcs $A(D) \setminus A'$ is the induced order by the one on $A(D)$. We write $s(D) := s_{A(D)}(D)$ for the resolution configuration obtained by **maximal surgery** of D along its arcs.

Definition. For two resolution configurations D and E, define $D \setminus E$ by

$$Z(D \setminus E) := Z(D) \setminus Z(E) \quad \text{and} \quad A(D \setminus E) := \{A \in A(D) | \forall Z \in Z(E) : A \cap Z = \emptyset\};$$

The total ordering on $A(D \setminus E)$ is induced by the one on $A(D)$.

The most important example of this construction is the case where $E = s_{A'}(D)$ for some $A' \subseteq A(D)$. Then $D \setminus E$ can be described as follows: $Z(D \setminus E)$ consists of those circles in D that are endpoints of arcs in A' and $A(D \setminus E) = A'$ with the induced ordering by $A(D)$. $D \setminus E$ has a useful property: It is a **basic resolution configuration** which means that every circle of this resolution configuration is an endpoint of an arc.

Definition. A **labelled resolution configuration** (D, x) is a resolution configuration D together with a labelling x of each circle $Z \in Z(D)$ by either x_- or x_+. We write $|x|$ for the "sum" of these labels, where x_+ counts as $+1$ and x_- as -1.

These labels obviously correspond to the generators x_- and x_+ of V in the first section. The generators $x_\pm \otimes \cdots \otimes x_\pm$ of $V^{\otimes c(v)}$ are in bijection to the labelling x of the circles in $Z(D_L(v))$ and the grading of such an element is precisely $|x|$. So for any $v \in \{0,1\}^n$, we can replace $V^{\otimes c(v)}$ in our notation by the graded free $\mathbb{Z}$-module generated by labelled resolution configurations $(D_L(v), x)$, where the grading of $(D_L(v), x)$ is given by $|x|$.

In order to define the boundary maps, we need to specify the image of each $(D_L(v), x)$. Along each boundary map, the homological grading increases by one, so if we write the image of $(D_L(v), x)$ in terms of elements $(D_L(u), y)$, then $|u| = |v| + 1$. Furthermore, we want the boundary maps to preserve (quantum) gradings, because the graded Euler characteristic of the chain complex should return the Jones polynomial. Hence, the quantum grading must decrease by 1 along each boundary map. (Remember the grading shift $\cdot\{|v|\}$ in the definition of $V(v)$ in Section 1.1.)

To take care of these two requirements, we define a partial order $\prec$ on labelled resolution configurations, so that the image of $(D_L(v), x)$ can be written in terms of those labelled resolution configurations "lying directly above" $(D_L(v), x)$ in the sense of the partial order $\prec$. We already have a natural partial order on the resolution configurations given by the partial order on $\{0,1\}^n$, where $v \leq u$ iff $v_i \leq u_i \, \forall i = 1, \ldots n$. We "refine" this partial order in the following definition.

Definition. The partial order $\prec$ on the set of labelled resolution configurations is defined as follows: Let (D, x) and (E, y) be two labelled resolution configurations. We write $(D, x) \prec_1 (E, y)$ iff

- (E, y) is obtained by performing an embedded surgery along a single arc A in $A(D)$,
- $x(Z) = y(Z)$ for all $Z \in Z(D) \setminus \{Z_1, Z_2\}$, where Z_1 and Z_2 are the (not necessarily distinct) endpoints of the arc A, and
- $|x| = |y| + 1$.

We write $(D, x) \prec (E, y)$ if $(D, x) = (E, y)$ or if there exists a chain

$$(D, x) \prec_1 \cdots \prec_1 (D_i, x_i) \prec_1 \cdots \prec_1 (E, y)$$

of labelled resolution configurations. Then $\prec$ is a well-defined partial order on the set of labelled resolution configurations, since the index of the labelled resolution configurations decrease by 1 along each $\prec_1$. We can consistently generalise the notation $\prec_1$ by writing $(D, x) \prec_i (E, y)$ where $i = \text{ind}(D) - \text{ind}(E)$.

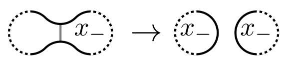
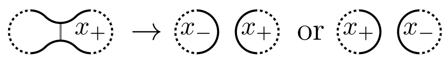
Definition. A **decorated resolution configuration** is a triple (D, x, y), where (D, y) is a resolution configuration and x is a labelling on $s(D)$ such that

$$(D, y) \prec (s(D), x).$$

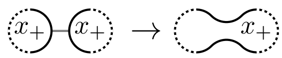
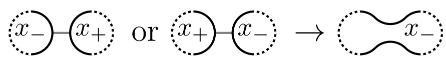
(D, x, y) is also called a decoration of D (or (D, y)). The poset of labelled resolution configurations (E, z) such that $(D, y) \prec (E, z) \prec (s(D), x)$ is denoted by $P(D, x, y)$.

Remark. Explicitly, the condition $|x| = |y| + 1$ in the definition of the partial order $\prec$ means the following:

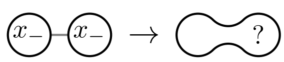
If $Z = Z_1 = Z_2$, let Z_1' and Z_2' denote the two circles in $s(D \setminus E)$. Then

- $x(Z) = x_- \Rightarrow y(Z_1') = x_- = y(Z_2')$: 
- $x(Z) = x_+ \Rightarrow \{y(Z_1'), y(Z_2')\} = \{x_+, x_-\}$: 

If $Z_1 \neq Z_2$, let Z' denote the single circle in $s(D \setminus E)$. Then

- $x(Z_1) = x_+ = x(Z_2) \Rightarrow y(Z') = x_+$: 
- $\{x(Z_1), x(Z_2)\} = \{x_-, x_+\} \Rightarrow y(Z') = x_-$: 

Also note that not every labelled resolution configuration can be decorated. Consider for example

.

1.3 Khovanov Homology

With the notion of labelled resolution configurations, it is not hard to define Khovanov homology.

Definition. We first define the **Khovanov chain complex** $KC^\bullet$ associated to a link diagram L. The chain module $KC^\bullet$ is a free $\mathbb{Z}$-module generated by the labelled resolution configurations $(D_L(u), x)$. This set of generators is denoted by $KG^\bullet$. We have two gradings on $KC^\bullet$, a quantum grading gr_q and a homological grading gr_h, defined by

$$\mathrm{gr}_q((D_L(u), x)) = n_+ - 2n_- + |u| + |x|$$
$$\mathrm{gr}_h((D_L(u), x)) = n_- + |u|.$$

The set of elements in homological grading i and quantum grading j is denoted by $KC^{i,j}$; likewise for the set of generators $KG^\bullet$. The boundary maps δ are defined by

$$\delta((D_L(v), y)) = \sum_{(D_L(v),y)\prec_1(D_L(u),x)} (-1)^{s(\mathcal{C}_{u,v})}(D_L(u), x),$$

where $s(\mathcal{C}_{u,v})$ is the sum over all entries of v up to the place where u and v differ. (The notation $s(\mathcal{C}_{u,v})$ will become clearer in Section 3.4.) Lemma 1 below says that this is a well-defined chain complex. Since the differential δ is grading preserving by construction, we get graded homology groups, the **Khovanov homology groups**

$$Kh^i(L) = \bigoplus_j Kh^{i,j}(L),$$

where $Kh^{i,j}(L)$ denotes the submodule of elements in $Kh^i(L)$ with quantum grading j.

One can show that the homology groups $Kh^{i,j}(L)$ are invariants of the (oriented) link, i. e. independent of the representing diagram L. For details, see [BN02] or Section 2.5, where we construct the Khovanov homotopy type and give an outline of the proof of invariance.

Lemma 1. δ is a boundary map, i. e. $\delta^2 = 0$.

Proof. Suppose $(D_L(v), y) \prec_2 (D_L(u), x)$ and let i and j be the places where v and u differ. There are exactly two resolution configurations between $D_L(v)$ and $D_L(u)$, namely $D_L(v_1)$ and $D_L(v_2)$, where v_1 and v_2 differ from v only in the places i and j respectively. We have

$$s(\mathcal{C}_{v_1,v}) + s(\mathcal{C}_{u,v_1}) + s(\mathcal{C}_{v_2,v}) + s(\mathcal{C}_{u,v_2}) \equiv 1 \mod 2. \quad (3)$$

So if we can show that we can label both $D_L(v_1)$ and $D_L(v_2)$ in the same number of ways, then the sign factor in the definition of the boundary map ensures that the $(D_L(u), x)$-component of $\delta^2((D_L(v), y))$ vanishes, i. e. $\delta^2 = 0$.
The labelling of the circles in $D_L(v)$ that are not endpoints of the two arcs along which we perform embedded surgery to get to $D_L(u)$ remains constant. It is therefore enough to consider the labelled resolution configurations that lie between $(D_L(v) \setminus D_L(u), y|)$ and $(s(D_L(v) \setminus D_L(u)), x|)$, where $y|$ and $x|$ denote the labellings induced by y and x, respectively. $D_L(v) \setminus D_L(u)$ is a basic index 2 resolution configuration. Since we need a description of resolution configurations of this type later on, we make our claim into the following lemma. ■

Lemma 2 (*Index 2 Lemma*)**.** For any basic decorated resolution configuration (D, x, y) of index 2, let E_1 and E_2 be the two resolution configurations between D and $s(D)$. Then

$$\#\{z_1|(D,y) \prec (E_1,z_1) \prec (s(D),x)\} = \#\{z_2|(D,y) \prec (E_1,z_2) \prec (s(D),x)\} \in \{1,2\}.$$

In fact, this integer is 2 iff D consists of one circle Z and two arcs A_1 and A_2 such that the four points in ∂A_1 and ∂A_2 alternate around the circle Z, see Figure 1 (h). We call such a resolution configuration a **ladybug configuration**.

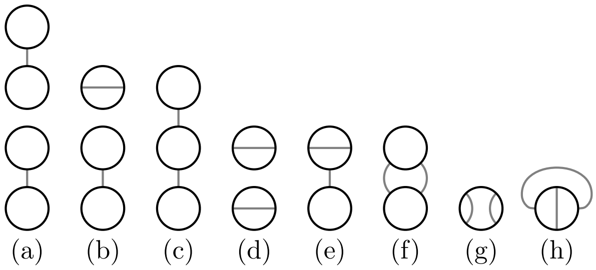


Figure 1: Basic index 2 resolution configurations

Proof. For the labelling of a resolution configuration and the partial order $\prec$, the relative positions of the arcs and circles do not matter. Up to this equivalence, Figure 1 shows a complete list of basic resolution configurations of index 2. We could verify the claim in each of these cases separately, which is not too difficult. Alternatively, we apply the Splitting Lemma of the next section (page 11) in most of these cases to get an isomorphism $P(D, x, y) \cong \{0, 1\}^2$ which respects the underlying cube structure. The remaining cases are (f) and (h). For (f), we can apply the n-arc Lemma (also on page 11). The last case (h), the ladybug configuration, is the only case where we need to check the lemma by hand: $(D, \{x_-\}, \{x_+\})$ is the only possible decoration of a ladybug configuration D; its corresponding poset $P(D, \{x_-\}, \{x_+\})$ is shown in Figure 2 below. ■

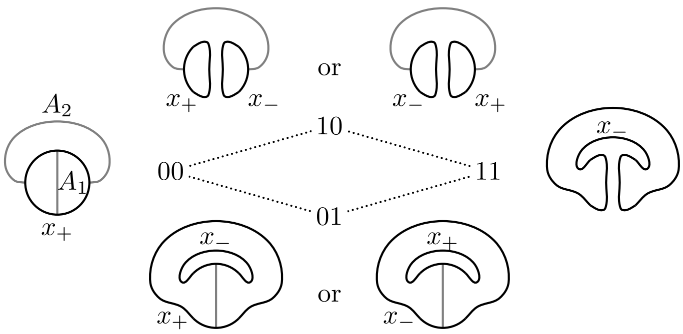
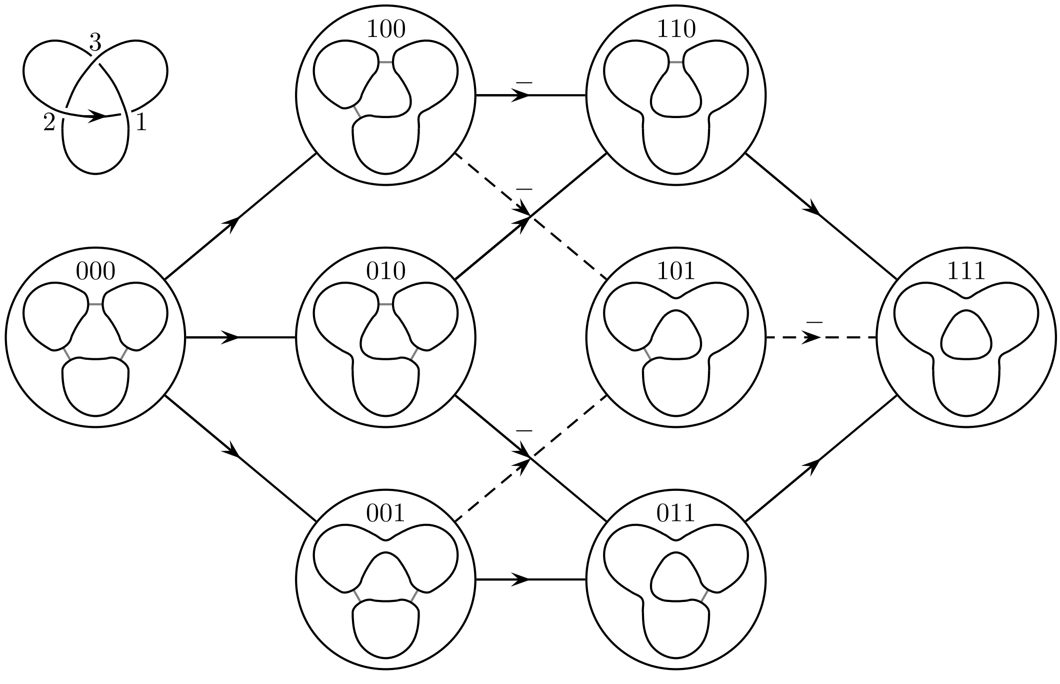


Figure 2: The poset of the decorated ladybug resolution configuration

Example. We want to compute the Khovanov homology invariants for one reasonably simple example, namely the left-handed trefoil knot. For this, we draw the resolution configurations to the corresponding vertices of the three-dimensional cube:



We also mark those differentials where $s(\mathcal{C}_{u,v})$ is odd with a minus sign. Note that $n_- = 3$, so the chain modules are concentrated in homological degrees 3 to 6 and the quantum grading of a labelled resolution configuration $(D_L(u), x)$ is given by $|x| + |u| - 6$. Choosing some ordering of the labelled resolution configurations, the chain complexes in the corresponding quantum gradings may be written in the following form:

$$-1\colon\quad 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}$$

$$-3\colon\quad \mathbb{Z} \xrightarrow{\begin{pmatrix} + \\ + \\ + \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} + & - & + \\ + & - & + \end{pmatrix}} \mathbb{Z}^2$$

$$-5\colon\quad \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} + & 0 & 0 \\ 0 & + & + \\ 0 & + & 0 \\ + & 0 & + \\ 0 & 0 & + \\ + & + & 0 \end{pmatrix}} \mathbb{Z}^6 \xrightarrow{\begin{pmatrix} - & - & + & + & 0 & 0 \\ - & - & 0 & 0 & + & + \\ 0 & 0 & - & - & + & + \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} + & - & + \end{pmatrix}} \mathbb{Z}$$

$$-7\colon\quad \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{pmatrix}} \mathbb{Z}^3 \longrightarrow 0 \longrightarrow 0$$

$$-9\colon\quad \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

Here, $+$ and $-$ are short for $+1$ and -1 respectively. After doing some elementary computations, we get

$$Kh^6 = \mathbb{Z}_{-1} \oplus \mathbb{Z}_{-3}, Kh^4 = \mathbb{Z}_{-5} \oplus (\mathbb{Z}/2)_{-7} \text{ and } Kh^3 = \mathbb{Z}_{-9},$$

where the subscripts denote the quantum gradings. Compare this to the corresponding unnormalised Jones polynomial:

$$J(\text{trefoil}) = q^{-1} + q^{-3} + q^{-5} - q^{-9}.$$

1.4 Decorated Resolution Configurations

In this section, we are going to examine decorated resolution configurations more closely. This is motivated by the following question.

> *Is the structure of the poset $P(D, x, y)$ of a decorated resolution configuration (D, x, y) already determined by just the resolution configuration D itself?*

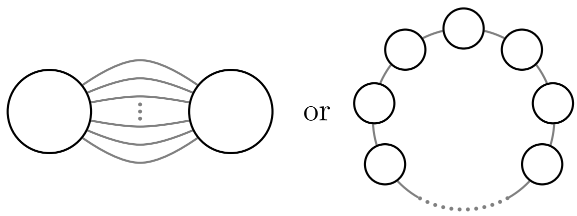
The following three lemmata are a partial answer to this question and they also give us simple descriptions of the posets. Apart from being quite interesting on their own, these lemmata are going to simplify the case analysis at the end of Section 2.3.

Definition. Given a resolution configuration D, we define its **dual** D^* by replacing every $\asymp$ by)(, i. e. we turn 0-resolutions into 1-resolutions $\asymp \to$) (and add arcs again. Note that $Z(D^*) = s(D)$ and $(D^*)^* = D$. Given a labelling y of D, its dual y^* is defined by $y^*(C) + y(C) = 0 \ \forall C \in Z(D)$. Thus, given a decorated resolution configuration (D, x, y), its dual can be defined as (D^*, y^*, x^*).

Lemma 1. $P(D, x, y) = P(D^*, y^*, x^*)$.

Proof. Check this for $\#A(D) = 1$ first and then apply induction. ■

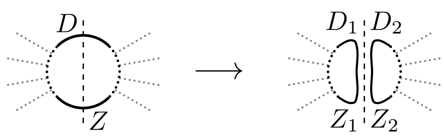
Lemma 2 (*n-arc Lemma*)**.** Let D be one of the following two resolution configurations with n arcs:



Then for any decoration (D, x, y) of D, we have $P(D, x, y) = \{0, 1\}^n$.

Proof. These two resolution configurations are dual to each other, so by Lemma 1 it is enough to prove the claim for the second one with n circles. Since $|y| = |x| + n$, there is at most one circle in D with labelling x_-. This is also true for every $(s_{A'}(D), z) \in P(D, x, y)$ with $A' \subsetneq A(D)$. Thus $P(D, x, y) = \{0, 1\}^n$. ■

Lemma 3 (*Splitting Lemma*)**.** Let D be a resolution configuration such that at some circle Z, it can be divided into two independent resolution configurations D_1 and D_2 like this:



Call the resulting circles Z_1 and Z_2 as shown above. Given a decoration (D, x, y) of D, consider some $(E, z) \in P(D, x, y)$. Then E also splits into two separate diagrams E_1 and E_2. Denote the circle in $Z(E)$ corresponding to Z by $c(E)$ and the circles in $Z(E_1)$ and $Z(E_2)$ corresponding to Z_1 and Z_2 by $c_1(E)$ and $c_2(E)$, respectively. Finally, define a labelling on E_1 and E_2. If $E = D$ and $C \in Z(D_i)$ $(i = 1, 2)$, let

$$y_i(C) = \begin{cases} y(C) & \text{if } C \neq Z_i \\ y(Z) & \text{if } C = Z_i \end{cases} . \qquad (4)$$

For arbitrary E and $C \in Z(E_i)$, let

$$z_i(C) = \begin{cases} z(C) & \text{if } C \neq c_i(E) \\ |y_i| - \operatorname{ind} D_i + \operatorname{ind} E_i - |z_i|_r & \text{if } C = c_i(E) \end{cases}, \tag{5}$$

where

$$|z_i|_r := \sum_{C \in Z(E_i) \smallsetminus \{c_i(E)\}} z_i(C).$$

See the remark below for motivation. We claim:

1. This labelling is well-defined and gives rise to well-defined decorated resolution configurations (D_1, x_1, y_1) and (D_2, x_2, y_2) along with a well-defined map

$$\begin{aligned} \varphi : P(D,x,y) &\to P(D_1,x_1,y_1) \times P(D_2,x_2,y_2) \\ (E,z) &\mapsto ((E_1,z_1),(E_2,z_2)). \end{aligned}$$

2. φ is an isomorphism of posets. Furthermore, this isomorphism is compatible with the underlying cube structure of the posets, i. e. we have a commutative diagram

$$\begin{array}{ccc} P(D,x,y) & \xrightarrow{\varphi} & P(D_1,x_1,y_1) \times P(D_2,x_2,y_2) \\ \downarrow & & \downarrow \\ \{0,1\}^{\operatorname{ind} D} & \xrightarrow{\sim} & \{0,1\}^{\operatorname{ind} D_1} \times \{0,1\}^{\operatorname{ind} D_2} \end{array}$$

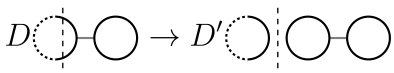
Remark. The labelling of $c_i(E)$ in definition (5) is such that

$$|z_i| = |y_i| - \operatorname{ind} D_i + \operatorname{ind} E_i,$$

which is a necessary condition for the map to be an isomorphism of posets. So the labellings on $Z(E_i) \smallsetminus \{c_i(E)\}$ already determine the labelling of the circles $c_i(E)$, but a priori it is not clear that $(|y_i| - \operatorname{ind} D_i + \operatorname{ind} E_i - |z_i|_r)$ is equal to ± 1. Note that (5) coincides with (4) for $(E,z) = (D,y)$, so we do not have to distinguish between these two cases. (But (4) is not redundant, because it defines $|y_i|$.)

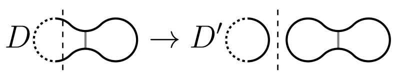
Example. Consider a resolution configuration D with a circle which is the endpoint of just one arc A, such that A has two distinct endpoints. Then the Splitting Lemma above tells us that a decoration (D,x,y) of D induces a decoration (D',x',y') of D' such that

$$P(D,x,y) \cong P(D',x',y') \times \{0,1\}$$



Similarly, we have

$$P(D,x,y) \cong P(D',x',y') \times \{0,1\}$$



Hence, by induction, the structure of $P(D,x,y)$ can be determined quite easily for many resolution configurations D. This is how we are going to apply the Splitting Lemma, so it serves exactly the same purpose as Lipshitz and Sarkar's leaf-lemma [LS11, Lemma 2.14]. However, it is a little more general and therefore perhaps a more satisfying result.

Proof of the Splitting Lemma.
Part 1. We show the first part of the lemma by an induction argument. We claim:

> If $z_i(c_i(E))$ is well-defined and $(E,z) \prec_1 (\tilde{E},\tilde{z})$ for some $(E,z),(\tilde{E},\tilde{z}) \in P(D,x,y)$, then $\tilde{z}_i(c_i(\tilde{E}))$ is also well-defined and $\varphi(E,z) \prec_1 \varphi(\tilde{E},\tilde{z})$. Furthermore,
> $$\text{if } z(c(E)) = x_+, \text{ then } z_i(c_i(E)) = x_+ \text{ for both } i = 1,2. \tag{$*$}$$

◀ Note that $z(c(E)) = x_-$ implies $\tilde{z}(c(\tilde{E})) = x_-$. The reason why we want to keep track of condition $(*)$ will become clear later. Now, assume wlog that $\tilde{E} = s_{\{A\}}(E)$ with $A \in A(D_1)$. Then clearly $\tilde{z}_2(c_2(\tilde{E})) = z_2(c_2(E))$ and $(E_2,z_2) = (\tilde{E}_2,\tilde{z}_2)$. In particular, $(*)$ is satisfied for $\tilde{E}$ and $i = 2$.
Next, we need to see that $\tilde{z}_1(c_1(\tilde{E}))$ is well-defined and that $(E_1,z_1) \prec_1 (\tilde{E}_1,\tilde{z}_1)$. We have

$$|z_1| = |y_1| - \operatorname{ind} D_1 + \operatorname{ind} E_1 = |y_1| - \operatorname{ind} D_1 + \operatorname{ind} \tilde{E}_1 + 1 = |\tilde{z}_1| + 1. \tag{6}$$

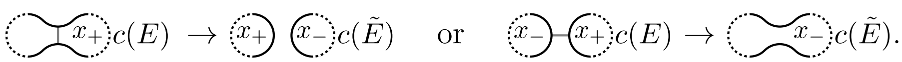
If $C \neq c_1(E)$ is not an endpoint of A, then $z_1(C) = \tilde{z}_1(C)$ by definition. If $c_1(E)$ is not an endpoint of A, then this is also true for $C = c_1(E) = c_1(\tilde{E})$:

$$\begin{aligned} z_1(c_1(E)) &= |y_1| - \operatorname{ind} D_1 + \operatorname{ind} E_1 - |z_1|_r \\ &= |y_1| - \operatorname{ind} D_1 + \operatorname{ind} \tilde{E}_1 + 1 - (|\tilde{z}_1|_r + 1) \;=\; \tilde{z}_1(c_1(\tilde{E})). \end{aligned} \tag{7}$$

In particular, $\tilde{z}_1(c_1(\tilde{E}))$ is then well-defined and $(*)$ holds for $\tilde{E}$ and $i = 1$. Finally, if $c_1(E)$ is an endpoint of A, then

$$z_1(c_1(E)) = |z_1| - |z_1|_r \overset{(6)}{=} |\tilde{z}_1| + 1 - |z_1|_r = \tilde{z}_1(c_1(\tilde{E})) + |\tilde{z}_1|_r - |z_1|_r + 1. \tag{8}$$

Since the number of circles changes by one and the labelling stays the same on those circles which are not endpoints of A, the value of $|\tilde{z}_1|_r - |z_1|_r + 1$ is either 0 or 2. If it is 0, then we are done and $(*)$ holds for $\tilde{E}$ and $i = 1$. If it is 2, then $|\tilde{z}_1|_r - |z_1|_r = 1$. A simple analysis of the list of possible steps on page 7 shows that we are in one of the following two cases:



So $z(c(E)) = x_+$, and by $(*)$, $z_1(c_1(E)) = x_+$; then $\tilde{z}_1(c_1(\tilde{E})) = x_-$, so it is well-defined and the hypothesis in $(*)$ is not satisfied for $\tilde{E}$, since $\tilde{z}(c(\tilde{E})) = x_-$. ▶

As mentioned above, we can now proceed by induction: For $(E,z) = (D,y)$, $(*)$ holds by definition. The (D_i,x_i,y_i) are indeed decorations of the (D_i,y_i), since we have a chain

$$(D,y) \prec_1 \cdots \prec_1 (E,z) \prec_1 \cdots \prec_1 (s(D),x),$$

which gives rise to two chains from (D_i,y_i) to $(s(D_i),x_i)$. By a similar argument, φ is well-defined and we have already checked that it respects the partial order.

Remark. The pairs $(y_1(Z_1), y_2(Z_2))$ and $(x_1(c_1(s(D))), x_2(c_2(s(D))))$ only differ in at most one place. In fact, it follows from the above that there are only four possibilities, namely

$$\begin{array}{llll}
\textbf{(a)} & (x_-, x_-) & \to & (x_-, x_-) \\
\textbf{(b)} & (x_+, x_+) & \to & (x_+, x_+) \\
\textbf{(c)} & (x_+, x_+) & \to & (x_-, x_+) \\
\textbf{(d)} & (x_+, x_+) & \to & (x_+, x_-).
\end{array}$$

Part 2. The commutative diagram condition is satisfied by construction. So it remains to show that the map is a bijection: We construct an inverse

$$\psi : ((E_1, z_1), (E_2, z_2)) \mapsto (E, z)$$

to φ, where for $C \in Z(E)$

$$z(C) = \begin{cases} z_1(C) & \text{for } C \in Z(E_1) \setminus \{c(E)\} \\ z_2(C) & \text{for } C \in Z(E_2) \setminus \{c(E)\} \\ |y| - \operatorname{ind} D + \operatorname{ind} E - |z|_r & \text{for } C = c(E) \end{cases} .$$

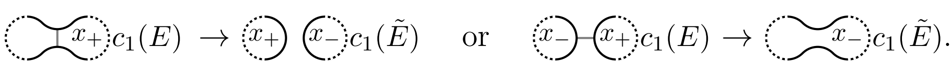
This is obviously an inverse, but we need to check that $z_i(c(E))$ and the map ψ itself are well-defined. Again, we apply an induction argument, similar to the one above. We show the following:

> Let $((E_1, z_1), (E_2, z_2)) \in P(D_1, x_1, y_1) \times P(D_2, x_2, y_2)$ and $(\tilde{E}_i, \tilde{z}_i) \in P(D_i, x_i, y_i)$ such that $(E_i, z_i) \prec_1 (\tilde{E}_i, \tilde{z}_i)$ for one i and equality for the other. Denote the image of $((\tilde{E}_1, \tilde{z}_1), (\tilde{E}_2, \tilde{z}_2))$ under ψ by $(\tilde{E}, \tilde{z})$. Then, if $z(c(E))$ is well-defined, so is $\tilde{z}(c(\tilde{E}))$ and we have $(E, z) \prec_1 (\tilde{E}, \tilde{z})$. Furthermore,
>
> $$\text{if } z_i(c_i(E)) = x_+ \text{ for both } i, \text{ then } z(c(E)) = x_+. \qquad (**)$$

◀ Again, it is clear that $|z| = |\tilde{z}| + 1$ by an argument analogous to (6). Wlog, let A be the arc in $A(E_1)$ such that $\tilde{E}_1 = s_{\{A\}}(E_1)$. If $C \in Z(E)$ with $C \neq c(E)$ is not an endpoint of A, then $z(C) = \tilde{z}(C)$ is also clear. If $c(E)$ is not an endpoint of A, then $z(c(E)) = \tilde{z}(c(\tilde{E}))$ by an equation similar to (7) and $(**)$ also holds. If $c(E)$ is an endpoint of A, by the same argument as in (8)

$$z(c(E)) = \tilde{z}(c(\tilde{E})) + |\tilde{z}|_r - |z|_r + 1.$$

Again, $|\tilde{z}|_r - |z|_r$ is either -1 or $+1$. We are done in the first case. In the second case, either



So we are in situation **(c)** (and **(d)**, if we swap the indices). Thus the hypothesis of $(**)$ is indeed satisfied, so $z(c(E)) = x_+$. Hence $\tilde{z}(c(\tilde{E})) = x_-$, so it is in particular well-defined. Since $\tilde{z}_1(c_1(\tilde{E})) = x_-$, we do not need to check $(**)$. ▶

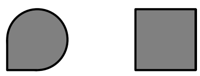
The condition $(**)$ is satisfied for $(E, z) = (D, y)$, so by induction $\tilde{z}(c(\tilde{E}))$ is indeed well-defined. By considering chains as above, we see that ψ is also well-defined. ■

2 A Khovanov Homotopy Type

Lipshitz and Sarkar's construction of a Khovanov homotopy type is split into two parts. In 1995, Cohen, Jones and Segal described a way of associating a CW complex (or more precisely: a suspension spectrum) to what is called a framed flow category [CJS95b]. Their work was motivated by the idea of generalising results in finite dimensional Morse theory to infinite settings [CJS95a]. In [LS11], Lipshitz and Sarkar used framed flow categories as the intermediate step on the way from link diagrams to "spaces".
The construction of a Khovanov flow category will occupy us for the Sections 2.1 to 2.3. In the remaining sections, we explain how to obtain a Khovanov homotopy type from a Khovanov flow category and sketch why this construction yields a link invariant.

2.1 $\langle n \rangle$-manifolds

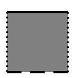
$\langle n \rangle$-manifolds are in some sense "nice" manifolds with corners plus some extra boundary structure. Recall that the definition of a manifold with corners is obtained by replacing the model space $\mathbb{R}^k$ in the usual definition of a smooth manifold by $(\mathbb{R}^+)^k = [0, \infty)^k$. As an example, consider the following two manifolds with corners:



Surely, we would not in general expect the boundary of such a manifold to be a manifold with corners again. In fact, this would be a restriction to manifolds with boundaries. However, as a compromise, it would be useful, if the "faces" of a manifold were manifolds with corners again. Let us make this idea precise (see [Jä68, p. 54]).

Definitions. Let X be a smooth k-dimensional manifold with corners. For any point $x \in X$, let $c(x)$ be the number of coordinates of x that are 0 with respect to a local chart containing x. One can show that this integer is well-defined (e. g. see [L03, p. 416]). The boundary of X is defined as $\partial X = \{x \in X | c(x) \geq 1\}$. The closure of a connected component of the codimension 1 boundary $\{x \in X | c(x) = 1\}$ is called a connected face. A face of X is the (possibly empty) union of pairwise disjoint connected faces of X.
X is called a **manifold with faces** if every point $x \in X$ belongs to exactly $c(x)$ connected faces. Not every manifold with corners is a manifold with faces – consider for example the one-dimensional disk with one corner above. However, now it is clear that every face of a k-dimensional manifold with faces is a $(k-1)$-dimensional manifold with faces.
It remains to define, what we mean by "extra boundary structure": An **$\langle n \rangle$-manifold** is obtained from a manifold with faces X by specifying an n-tuple $(\partial_1 X, \ldots, \partial_n X)$ of faces of X such that every point $x \in X$ belongs to exactly $c(x)$ of these faces $\partial_i X$. In other words, every connected face of X belongs to exactly one of the faces $\partial_i X$. By convention, if $n \leq 0$, the n-tuple will be empty.

Remark. The number of faces n is independent of the dimension k. Consider for example the following $\langle 3 \rangle$-manifold structure on the square:



Given a k-dimensional $\langle n\rangle$-manifold X, the $\partial_i X's$ are $(k-1)$-dimensional $\langle n-1\rangle$-manifolds with faces given by $(\partial_i X \cap \partial_j X)_{j\neq i}$. (Such a face is empty if $k \leq 0$ or $n \leq 0$.) This observation gives rise to another point of view on $\langle n\rangle$-manifolds: For every $l > 0$, we inductively get $(k-l)$-dimensional $\langle n-l\rangle$-manifolds $\partial_{i_1} X \cap \cdots \cap \partial_{i_{l-1}} X$. Then, for every element $v \in \{0,1\}^n$, we set

$$X(v) := \begin{cases} X & \text{for } v = \underline{1} \\ \bigcap_{v_i=0} \partial_i X & \text{otherwise} \end{cases} .$$

Furthermore, if $v \leq u$ in $\{0,1\}^n$, we have the obvious inclusions $X(v) \hookrightarrow X(u)$, so we can regard X as a functor from the partial order category $\{0,1\}^n$ to the category of topological spaces. We call such a functor an n-**diagram**. From this viewpoint, we can define the **product of** $\langle n\rangle$**-manifolds**:

Definition. The product of an $\langle n\rangle$-manifold X and an $\langle m\rangle$-manifold Y is the product functor

$$\{0,1\}^{n+m} = \{0,1\}^n \times \{0,1\}^m \to X \times Y.$$

Then the $\langle n+m\rangle$-manifold structure on $X \times Y$ is given by

$$\partial_i(X \times Y) = \begin{cases} \partial_i X \times Y & \text{for } i \leq n \\ X \times \partial_{i-n} Y & \text{for } i > n \end{cases} . \tag{1}$$

Definition. A smooth map $f : X \to Y$ between two $\langle n\rangle$-manifolds X and Y is called an n-**map** if $f^{-1}(\partial_i Y) = \partial_i X$.

Remark. If $f : X \to Y$ is such an n-map, then $f^{-1}(\partial_i Y) = \partial_i X$ implies that for any $v \in \{0,1\}^n$, we can restrict f to $f(v) := f|_{X(v)} : X(v) \to Y(v)$ and this is a $|v|$-map. So viewing $\langle n\rangle$-manifolds as n-diagrams, an n-map can be seen as a special kind of natural transformation between two $\langle n\rangle$-manifolds.

In the construction of the Khovanov flow category, we will define n-dimensional $\langle n\rangle$-manifolds by induction on n. In the induction step, we need a way of glueing the $\langle n\rangle$-manifolds that we have already defined together to give the boundary of the $\langle n+1\rangle$-manifold that we want to construct. For this we introduce the concept of n-boundaries:

Definition. Delete the topmost vertex $\underline{1}$ in $\{0,1\}^n$ and denote the full subcategory thus obtained by $\{0,1\}^n \setminus \underline{1}$. A functor from $\{0,1\}^n \setminus \underline{1}$ to the category of topological spaces is called a truncated n-diagram. A truncated n-diagram X is called a k**-dimensional** n**-boundary** if the restriction of X to the full subcategory $\{v \in \{0,1\}^n | v_i = 0\}$ of $\{0,1\}^n \setminus \underline{1}$ is a k-dimensional $\langle n-1\rangle$-manifold for all $i = 1,\ldots,n$.

Note that given an n-boundary X, its restriction to the full subcategory

$$\{u \in \{0,1\}^n | u \leq v\} \subseteq \{0,1\}^n \setminus \underline{1} \quad \text{for any } v \in \{0,1\}^n \setminus \underline{1}$$

is an $\langle n-l\rangle$-manifold, which we denote by $X(v)$. Also, the maps $X(v_1) \to X(v_2)$ are embeddings for $v_1 < v_2$ in $\{0,1\}^n \setminus \underline{1}$.

We can then form the colimit by the usual construction

$$\operatorname{Colim} X = \left(\coprod_{v \in \{0,1\}^n \smallsetminus \underline{1}} X(v) \right) / \sim,$$

where we identify two points $x_1 \in X(v_1)$ and $x_2 \in X(v_2)$ if there is some $w \in \{0,1\}^n \smallsetminus \underline{1}$ and some $y \in X(w)$ such that $w < v_i$ and the inclusion $X(w) \to X(v_i)$ sends y to x_i for both $i = 1, 2$.
Observe that if we restrict an $\langle n \rangle$-manifold X to $\{0,1\}^n \smallsetminus \underline{1}$, we obtain an n-boundary $X|$ such that $\operatorname{Colim} X|$ is exactly the boundary ∂X of X, hence the name "n-boundary".

We conclude this subsection with a lemma which we need later in the induction step of the construction of the Khovanov flow category.

Lemma. Let X and Y be two k-dimensional n-boundaries and $F : X \to Y$ a natural transformation such that the smooth maps $F(v) : X(v) \to Y(v)$ are coverings of $Y(v)$ and $|v|$-maps for all topmost $v \in \{0,1\}^n \smallsetminus \underline{1}$ (and hence for the rest, too). Then the induced map $\partial F := \operatorname{colim} F : \operatorname{Colim} X \to \operatorname{Colim} Y$ is also a covering.

Proof. Take any $y \in \operatorname{Colim} Y$. Let w be the (unique) minimal vector in $\{0,1\}^n$ such that $y \in Y(v)$. Let $x \in (\partial F)^{-1}(y)$. Then, using the fact that the $F(v)$ are $|v|$-maps and F is natural, it is not hard to see that w is also the minimal vector in $\{0,1\}^n$ such that $x \in X(w)$. Let $v^1, \ldots, v^d \in \{0,1\}^n$ be the vectors above w with $|v^i| = n-1$. These are exactly those vectors such that $x \in X(v^i)$ (and the same for y). Note that $d = n - |w|$. This integer is called the depth of a point x in an $\langle n \rangle$-boundary, see [LS11, p. 12].
For all $i = 1, \ldots, k$, take a small neighbourhood $V_i \subset Y(v^i)$ of y and $U_i \subset X(v^i)$ of x such that U_i and V_i are homeomorphic via $F(v^i)$. Let U be an open neighbourhood of x contained in $U_1 \cup \cdots \cup U_k$ and $U_i' := U \cap U_i$. Let V_i' be the images of U_i' under the homeomorphisms $U_i \cong V_i$. Then $V := V_1' \cup \cdots \cup V_k'$ is an open neighbourhood of y in $\operatorname{Colim} Y$ such that U is homeomorphic to V via ∂F. ■

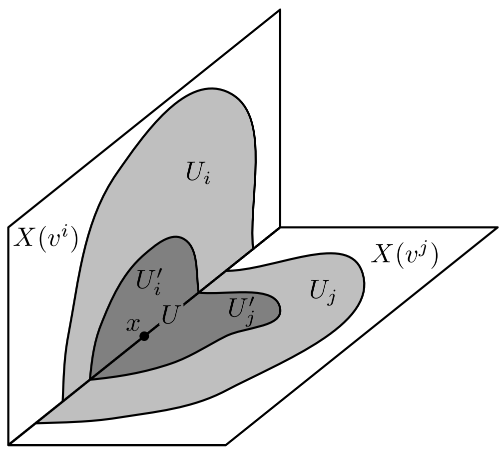


Figure 3: Illustration of the proof above

2.2 The Cube Flow Category $\mathscr{C}_C$

Definition. A flow category $\mathscr{C}$ is a category with finitely many objects together with a grading $\mathrm{gr} : \mathrm{ob}\mathscr{C} \to \mathbb{Z}$ on objects such that the following conditions are satisfied. We write $\mathbf{x} \succ_m \mathbf{y}$ for $\mathrm{gr}(\mathbf{x}) = \mathrm{gr}(\mathbf{y}) + m$, where $\mathbf{x}, \mathbf{y} \in \mathrm{ob}\,\mathscr{C}$ and $m \geq 0$.

(FC-1) Each set of morphisms $\mathcal{M}(\mathbf{x}, \mathbf{y})$, also called **moduli space**, between two distinct objects $\mathbf{x}$ and $\mathbf{y}$ carries the structure of a compact $(\mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\mathbf{y}) - 1)$-dimensional $\langle \mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\mathbf{y}) - 1 \rangle$-manifold. Here, a negative dimensional manifold is considered to be empty. We set $\mathcal{M}(\mathbf{x}, \mathbf{x}) := \{id\}$ for each object $\mathbf{x} \in \mathrm{ob}\mathscr{C}$.

(FC-2) The composition

$$\circ : \mathcal{M}(\mathbf{z}, \mathbf{y}) \times \mathcal{M}(\mathbf{x}, \mathbf{z}) \to \mathcal{M}(\mathbf{x}, \mathbf{y}),$$

of morphisms $\mathbf{x} \to \mathbf{z} \to \mathbf{y}$ is an embedding into $\partial_m \mathcal{M}(\mathbf{x}, \mathbf{y})$, where $\mathbf{z} \succ_m \mathbf{y}$.
It satisfies

$$\circ^{-1}(\partial_i \mathcal{M}(\mathbf{x}, \mathbf{y})) = \begin{cases} \partial_i \mathcal{M}(\mathbf{z}, \mathbf{y}) \times \mathcal{M}(\mathbf{x}, \mathbf{z}) & \text{for } i < m \\ \mathcal{M}(\mathbf{z}, \mathbf{y}) \times \partial_{i-m} \mathcal{M}(\mathbf{x}, \mathbf{z}) & \text{for } i > m \end{cases} .$$

Compare this to the identity (1) in the previous section. Also note that for $\mathbf{x} = \mathbf{z}$ or $\mathbf{z} = \mathbf{y}$, $\circ$ is just the identity on the other factor.

(FC-3) The $(\mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\mathbf{y}) - 1)$ faces of each $\mathcal{M}(\mathbf{x}, \mathbf{y})$ are of the form

$$\partial_i \mathcal{M}(\mathbf{x}, \mathbf{y}) = \coprod_{\mathbf{z} \succ_i \mathbf{y}} \circ(\mathcal{M}(\mathbf{z}, \mathbf{y}) \times \mathcal{M}(\mathbf{x}, \mathbf{z})).$$

Motivation. We want to motivate this definition by looking at how the notion of flow categories arises in Morse theory. In Morse theory, one studies Riemannian manifolds M by considering the critical points of real valued smooth functions f on them. If f has non-degenerate critical points, i. e. the Hessian of f at each critical point is non-singular, then it is called a **Morse function**. Given such a Morse function f on the manifold M, we can assign a non-negative integer to every critical point of f: The index $\mathrm{ind}(p)$ of a critical point p can be defined as the number of negative eigenvalues of the Hessian matrix at p with respect to some local coordinates.
As illustrated in Figure 4 (a), we can think of f as the height function on the manifold M (embedded into some $\mathbb{R}^n$) with respect to some hyperplane. In the example of this "sausage-shaped" sphere, we have four critical points: $\mathbf{x}$ and $\mathbf{x}'$ both have index 2, $\mathbf{z}$ and $\mathbf{y}$ have index 1 and 0, respectively. The critical points will become the objects of the flow category $\mathscr{C}_f$ associated with f and the index will give the grading on objects.
Where do the moduli spaces $\mathcal{M}(\cdot, \cdot)$ of the flow category $\mathscr{C}_f$ come from? For this, we consider smooth paths on M whose tangent vectors coincide with the negative gradient vector field induced by f, i. e. paths which are the unique solutions $\gamma = \gamma(t)$ of the differential equation

$$\frac{d\gamma}{dt} = -\mathrm{grad} f(\gamma(t)).$$

They are called **flow lines**. Suppose, M is compact. Then one can show that for $t \to \pm\infty$, $\gamma(t)$ converges to some critical point (see [N00, Section 2.4]). Figure 4 (a) shows some flow lines, for example the two dashed ones from $\mathbf{x}$ to $\mathbf{y}$.

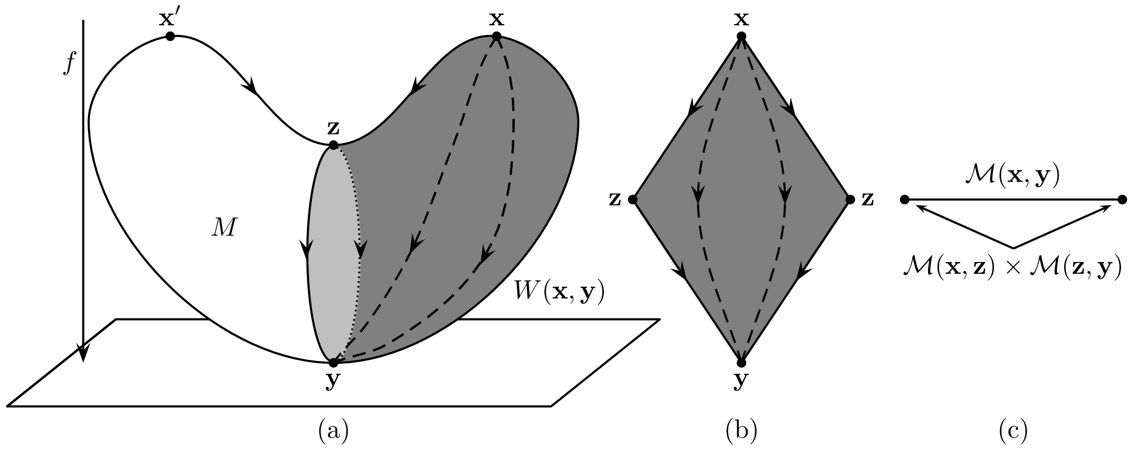


Figure 4: (a) A Morse function and flow lines, (b)–(c) a moduli space $\mathcal{M}(\mathbf{x}, \mathbf{y})$

This observation allows us to consider the following subsets

$$W^-(\mathbf{x}) := \{p \in M | p \text{ lies on a flow line } \gamma \text{ with } \lim_{t \to -\infty} \gamma(t) = \mathbf{x}\} \subseteq M$$

and

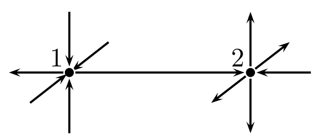
$$W^+(\mathbf{y}) = \{p \in M | p \text{ lies on a flow line } \gamma \text{ with } \lim_{t \to \infty} \gamma(t) = \mathbf{y}\} \subseteq M$$

for some critical points $\mathbf{x}$ and $\mathbf{y}$ on our compact manifold M. A standard result in Morse theory is that $W^-(\mathbf{x})$ and $W^+(\mathbf{y})$ are manifolds, in fact, they are diffeomorphic to open discs of dimension $\text{ind}(\mathbf{x})$ and $(\dim M - \text{ind}(\mathbf{y}))$ respectively [N00, Section 2.4]. $W^-(\mathbf{x})$ is called the **unstable manifold** of $\mathbf{x}$ and $W^+(\mathbf{y})$ is called the **stable manifold** of $\mathbf{y}$. In Figure 4 (a), $W^-(\mathbf{x})$ consists of the grey part (without the two flow lines from $\mathbf{z}$ to $\mathbf{y}$ and the one from $\mathbf{x}$ to $\mathbf{z}$); $W^+(\mathbf{y})$ is all of M with the exceptions of those points lying on the flow lines from $\mathbf{x}'$ to $\mathbf{z}$ and from $\mathbf{x}$ to $\mathbf{z}$ (and the critical points $\mathbf{x}$, $\mathbf{x}'$ and $\mathbf{z}$).
A Morse function is said to satisfy the **Morse-Smale** condition if the intersection of the unstable and stable manifolds corresponding to any pair of critical points $(\mathbf{x}, \mathbf{y})$ is transversal. Then,

$$W(\mathbf{x}, \mathbf{y}) := W^-(\mathbf{x}) \cap W^+(\mathbf{y})$$

is a manifold of dimension $\text{ind}(\mathbf{x}) - \text{ind}(\mathbf{y})$. In particular, if $\mathbf{y} \succeq \mathbf{x}$ (i. e. $\text{ind}(\mathbf{y}) \geq \text{ind}(\mathbf{x})$) and $\mathbf{x} \neq \mathbf{y}$, then $W(\mathbf{x}, \mathbf{y})$ is empty, since if $W(\mathbf{x}, \mathbf{y})$ is nonempty, it contains at least one flow line, which is one-dimensional for $\mathbf{x} \neq \mathbf{y}$.

Remark. This transversality condition does not hold in general. However, by "perturbing" any Morse functions, we can obtain a Morse function satisfying the Morse-Smale condition (see [N00, p. 57]). For a good visualisation of this result, see the example of the height function on the torus at [WikiMS]. It is also not hard to draw examples with flow lines from $\mathbf{x}$ to $\mathbf{y}$ where $\text{ind}(\mathbf{x}) < \text{ind}(\mathbf{y})$:



Suppose now, f is a Morse function satisfying the Morse-Smale condition. In order to define the moduli spaces $\mathcal{M}(\mathbf{x},\mathbf{y})$, we also have to take care of the boundary of $W(\mathbf{x},\mathbf{y})$, consisting of so-called broken flow lines. We make this precise.
First, we reparametrize the flow lines γ from $\mathbf{x}$ to $\mathbf{y}$ and add their endpoints such that

$$\gamma : [f(\mathbf{y}), f(\mathbf{x})] \to M \text{ with } f(\gamma(t)) = |t|, \gamma(f(\mathbf{y})) = \mathbf{y} \text{ and } \gamma(f(\mathbf{x})) = \mathbf{x}. \tag{2}$$

Secondly, we define a broken flow line from $\mathbf{x}$ to $\mathbf{y}$ to be a path which is the result of glueing several flow lines together: Given flow lines $\gamma_i : [f(\mathbf{z}^i), f(\mathbf{z}^{i-1})] \to M$ for critical points $\mathbf{z}^i \in M, i = 0, \dots, m \geq 2$ such that

$$\mathbf{y} = \mathbf{z}^0 \prec \mathbf{z}^1 \prec \dots \prec \mathbf{z}^{m-1} \prec \mathbf{z}^m = \mathbf{x},$$

define the corresponding broken flow line $\gamma : [f(\mathbf{y}), f(\mathbf{x})] \to M$ by

$$\gamma(t) := \gamma_i(t) \text{ if } t \in [f(\mathbf{z}^i), f(\mathbf{z}^{i-1})].$$

Then define $\mathcal{M}(\mathbf{x},\mathbf{y})$ to be the set of all flow lines and broken flow lines from $\mathbf{x}$ to $\mathbf{y}$. In particular, $\mathcal{M}(\mathbf{x},\mathbf{x})$ is just the one-element set consisting of the constant path. This definition is illustrated in Figure 4 (b)–(c).

Theorem. Let M be a compact Riemannian manifold and f a Morse function satisfying the Morse-Smale condition. Then the sets $\mathcal{M}(\cdot,\cdot)$ constructed above can be given the structure of $\langle n \rangle$-manifolds. Moreover, they are the moduli spaces of a flow category $\mathscr{C}_f$, whose objects are the critical points of f and the grading is the index function on critical points.

Remark. Suppose, we are given such a flow category which is the result of a decomposition of a Riemanian manifold as described above using a Morse function. Cohen, Jones and Segal showed that we can recover the manifold by a glueing process (see [CJS95a] or [HD10]). For an arbitrary flow category, they described a different construction ([CJS95b, Section 5, pp. 308ff]) which Lipshitz and Sarkar used in [LS11] for the construction of the Khovanov homotopy type (see Section 2.4).

Example. Consider the Morse function $f(x) = 3x^2 - 2x^3$ on $\mathbb{R}$. f has two critical points, namely one of index 0 at 0 and one of index 1 at 1. More generally, consider the function

$$f_n(x_1, \dots, x_n) := f(x_1) + \dots + f(x_n)$$

on $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Its critical points correspond to the vertices of the cube $[0,1]^n$ and for each such vertex $v \in \{0,1\}^n$, its index is given by $|v| = \sum_i v_i$. The above theorem gives us a corresponding flow category $\mathscr{C}_{f_n}$, the n-dimensional **cube flow category**. We write $\mathscr{C}_C(n)$ for $\mathscr{C}_{f_n}$, or $\mathscr{C}_C$ for short. ($\mathbb{R}^n$ being non-compact does not matter, because we are only interested in what happens in the cube $[0,1]^n$.)

The proof of the above theorem requires some careful analysis of the broken flow lines (see [CJS95a] or [CJS95b, Section 3, pp. 302ff]) and it would probably be overkill for our needs. Therefore, we give an alternative construction of $\mathscr{C}_C$ without using a Morse function. This also has the advantage of obtaining a more detailed description of the boundaries of the moduli spaces which will prove to be useful later on in the next section.

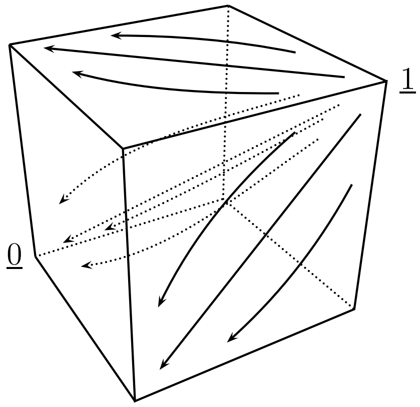


Figure 5: The three-dimensional cube with some (broken) flow lines

Proof of the theorem for $\mathscr{C}_C$. We define the cube flow categories $\mathscr{C}_C(n)$ inductively. Write $\mathcal{M}_n(\cdot,\cdot)$ for the moduli spaces $\mathcal{M}_{\mathscr{C}_C(n)}(\cdot,\cdot)$. Let $\mathscr{C}_C(1)$ be the flow category with exactly two objects 0 and 1 with the obvious grading and $\mathcal{M}_1(1,0)$ being a single point. Suppose, we have constructed the n-dimensional cube flow category $\mathscr{C}_C(n)$. The objects of $\mathscr{C}_C(n{+}1)$ are given by $\{0,1\}^{n+1}$ with the obvious poset structure and grading. For each pair (u,v) of objects $u,v \in \{0,1\}^{n+1}$ with $u > v$ and $(u,v) \neq (\underline{1},\underline{0})$, we define $\mathcal{M}_{n+1}(u,v) := \mathcal{M}_{|u|-|v|}(\underline{1},\underline{0})$ as $\langle |u|-|v|-1\rangle$-manifolds and write

$$\mathcal{I}_{u,v} : \mathcal{M}_{|u|-|v|}(\underline{1},\underline{0}) \to \mathcal{M}_{n+1}(u,v)$$

for this identification. Then for any w with $v <_m w < u$, we define

$$\circ : \mathcal{M}_{n+1}(w,v) \times \mathcal{M}_{n+1}(u,w) \to \mathcal{M}_{n+1}(u,v)$$

via the commutative diagram

$$\begin{array}{ccc}
\circ : \mathcal{M}_{n+1}(w,v) \times \mathcal{M}_{n+1}(u,w) & \longrightarrow & \mathcal{M}_{n+1}(u,v) \\
\mathcal{I}_{w,v}\times\mathcal{I}_{u,w}\uparrow & & \uparrow\mathcal{I}_{u,v} \\
\mathcal{M}_{|w|-|v|}(\underline{1},\underline{0}) \times \mathcal{M}_{|u|-|w|}(\underline{1},\underline{0}) & \longrightarrow & \mathcal{M}_{|u|-|v|}(\underline{1},\underline{0}) \\
\mathcal{I}_{w',\underline{0}}\times\mathcal{I}_{\underline{1},w'}\downarrow & \nearrow\circ & \\
\mathcal{M}_{|u|-|v|}(w',\underline{0}) \times \mathcal{M}_{|u|-|v|}(\underline{1},w') & &
\end{array}$$

where w' is the vector obtained from w by deleting all entries where u and v agree.
Then for $(u,v) \neq (\underline{1},\underline{0})$, condition (FC-1) is obviously satisfied. Also, $\circ$ is an embedding into $\partial_m\mathcal{M}_{n+1}(u,v)$. The second part of condition (FC-2) follows easily from the above diagram and so does condition (FC-3). Now it remains to define $\mathcal{M}_{n+1}(\underline{1},\underline{0})$ as well as compositions into $\mathcal{M}_{n+1}(\underline{1},\underline{0})$ and to show (FC-2) and (FC-3) for the remaining cases.

For the definition of $\mathcal{M}_{n+1}(\underline{1},\underline{0})$, we first consider the following diagram $\mathfrak{C}$: Each sequence

$$\underline{1} > v^{(r)} > \cdots > v^{(1)} > \underline{0}, \quad r \geq 1 \tag{3}$$

corresponds to a vertex

$$\mathfrak{C}(v^{(r)}, \ldots, v^{(1)}) := \mathcal{M}_{n+1}(v^{(1)}, \underline{0}) \times \cdots \times \mathcal{M}_{n+1}(\underline{1}, v^{(r)})$$

in $\mathfrak{C}$. The arrows of $\mathfrak{C}$ are given by composing adjacent pairs of moduli spaces. Next, we define an n-boundary by taking disjoint unions over certain subsets of these vertices. To each sequence of the form (3), we can associate a vector $v = \mathfrak{v}(v^{(r)}, \ldots, v^{(1)}) \in \{0,1\}^n \setminus \underline{1}$ by setting

$$v_i = 0 \Leftrightarrow \exists j : |v^{(j)}| = i.$$

We now take the disjoint union over those vertices which give the same v, i. e. we define

$$B_C(v) = \coprod_{v=\mathfrak{v}(v^{(r)},\ldots,v^{(1)})} \mathfrak{C}(v^{(r)}, \ldots, v^{(1)})$$

together with the induced composition maps from $\mathfrak{C}$. Then $B_C(v)$ is a disjoint union of some $|v|$-dimensional $\langle |v| \rangle$-manifolds. In particular for $|v| = n-1$ with $v_m = 0$, we have

$$B_C(v) = \coprod_{v=\mathfrak{v}(v')} \mathfrak{C}(v') = \coprod_{|v'|=m} \mathcal{M}_{n+1}(v', \underline{0}) \times \mathcal{M}_{n+1}(\underline{1}, v'),$$

so it is natural to define

$$\partial_m \mathcal{M}_{n+1}(\underline{1},\underline{0}) := B_C(v) \quad \text{and} \quad \operatorname{Colim} B_C =: \partial \mathcal{M}_{n+1}(\underline{1},\underline{0}).$$

We can define the compositions into $\mathcal{M}_{n+1}(\underline{1},\underline{0})$ to be given by the obvious inclusions:

$$\circ : \mathcal{M}_{n+1}(v,\underline{0}) \times \mathcal{M}_{n+1}(\underline{1}, v) \hookrightarrow \mathcal{M}_{n+1}(\underline{1},\underline{0}).$$

Hence, (FC-3) is true for $\mathcal{M}_{n+1}(\underline{1},\underline{0})$ by definition. For the second part of (FC-2), we need to consider $\circ^{-1}(\partial_i \mathcal{M}_{n+1}(\underline{1},\underline{0}))$, which we can write as

$$\mathcal{M}_{n+1}(v,\underline{0}) \times \mathcal{M}_{n+1}(\underline{1}, v) \cap \coprod_{|u|=i} \mathcal{M}_{n+1}(u,\underline{0}) \times \mathcal{M}_{n+1}(\underline{1}, u) \subseteq \operatorname{Colim} B_C.$$

It is clear from the construction that we only need to consider those u which lie below v (if $i < m$) or above v (if $i > m$). Thus, we obtain

$$\begin{aligned}
&\mathcal{M}_{n+1}(v,\underline{0}) \times \mathcal{M}_{n+1}(\underline{1}, v) \cap \coprod_{\substack{|u|=i \\ u<v \text{ or } v>v}} \mathcal{M}_{n+1}(u,\underline{0}) \times \mathcal{M}_{n+1}(\underline{1}, u) \\
&= \coprod_{\substack{|u|=i \\ u<v \text{ or } v>v}} \begin{cases} \mathcal{M}_{n+1}(u,\underline{0}) \times \mathcal{M}_{n+1}(v,u) \times \mathcal{M}_{n+1}(\underline{1}, v) & \text{for } i < m \\ \mathcal{M}_{n+1}(v,\underline{0}) \times \mathcal{M}_{n+1}(u,v) \times \mathcal{M}_{n+1}(\underline{1}, u) & \text{for } i > m \end{cases} \\
&\overset{(*)}{=} \begin{cases} \partial_i \mathcal{M}_{n+1}(v,\underline{0}) \times \mathcal{M}_{n+1}(\underline{1}, v) & \text{for } i < m \\ \mathcal{M}_{n+1}(v,\underline{0}) \times \partial_{i-m} \mathcal{M}_{n+1}(\underline{1}, v) & \text{for } i > m \end{cases}
\end{aligned}$$

In the last step $(*)$, we use the identifications $\mathcal{I}_{\cdot}$ and the induction hypothesis. So we have a well-defined boundary of $\mathcal{M}_{n+1}(\underline{1},\underline{0})$. It remains to "'fill it up". This is taken care of by the following lemma. ■

Lemma. $\text{Colim}B_C$ is homotopic to S^{n-1} and we can define $\mathcal{M}_{n+1}(\underline{1},\underline{0})$ to be the disk D^n with a well-defined $\langle n\rangle$-manifold structure induced by the n-boundary structure on S^{n-1}.

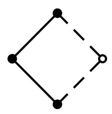
Before we come to the proof of this lemma, we describe the cube flow category in low dimensions to illustrate the constructions in the proof above. For $n = 2$, the diagram $\mathfrak{C}$ consists of exactly two vertices, corresponding to the two chains

$$\underline{1} > (0,1) > \underline{0} \quad \text{and} \quad \underline{1} > (1,0) > \underline{0}.$$

Then the boundary B_C consists of the single vertex $B_C(\underline{0})$, which is just two points. We "fill it up" to get an interval, which gives us $\mathcal{M}_{\mathscr{C}_C(2)}(\underline{1},\underline{0})$.
Next, we want to work out what $\mathscr{C}_C(3)$ looks like. $\mathfrak{C}$ then has six vertices corresponding to chains of the form $\underline{1} > v^{(1)} > \underline{0}$, each of which gives one interval, and another six of the form $\underline{1} > v^{(2)} > v^{(1)} > \underline{0}$, each of which corresponds to a point. The intervals are of two types. They either correspond to

$$\underline{1} >_2 v^{(1)} >_1 \underline{0} \quad \text{or} \quad \underline{1} >_1 v^{(1)} >_2 \underline{0}.$$

The 2-boundary distinguishes between these two types – the disjoint union of the first type is $B_C((0,1)) = \partial_1\mathcal{M}_{\mathscr{C}_C(3)}(\underline{1},\underline{0})$ and the others form $B_C((1,0)) = \partial_2\mathcal{M}_{\mathscr{C}_C(3)}(\underline{1},\underline{0})$. The diagram of B_C looks like this:



In the colimit of B_C (indicated by the dot $\circ$ in the diagram above), these two faces are glued together via the inclusions $B_C(\underline{0}) \hookrightarrow B_C((1,0))$ and $B_C(\underline{0}) \hookrightarrow B_C((0,1))$. It is clear from the construction of these maps that each point in $B_C(\underline{0})$ belongs to a face of exactly one interval in both $B_C((0,1))$ and $B_C((1,0))$, such that they glue together to form a hexagon. Figure 6 shows this hexagon in the 3-dimensional cube. The first face of $\mathcal{M}_{\mathscr{C}_C(3)}(\underline{1},\underline{0})$ is highlighted.

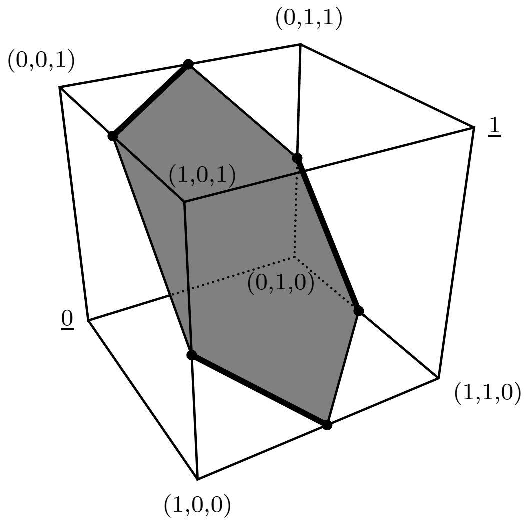


Figure 6: Illustration of the 3-dimensional cube flow category

In the 4-dimensional case, we have four hexagons in both $B_C((1,1,0))$ and $B_C((0,1,1))$ and six squares in $B_C(1,0,1)$, which glue together to give the boundary of the polytope in Figure 7.

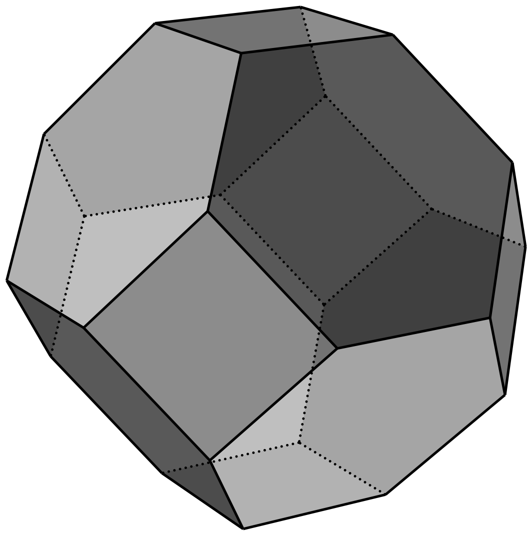


Figure 7: Illustration of the $\langle 3\rangle$-manifold structure on $\mathcal{M}_4(\underline{1},\underline{0})$

We now come to the proof of the lemma. For this, we go through the above construction again, specifying the moduli spaces as topological spaces. This description originates from considering some kind of "product construction" for flow categories.

Proof of the lemma. For any vector $v = (v_1, \ldots, v_{n+1})$, write v^i for the truncated vector $(v_1, \ldots, v_i)$. For $u, v \in \{0,1\}^{n+1}$, $u > v$, we define

$$\mathcal{M}_{n+1}(u,v) := \bigtimes_{i=1}^{n} \mathcal{N}_i(u,v),$$

where

$$[0,i] \supseteq \mathcal{N}_i(u,v) = \begin{cases} [|v^i|, |u^i|] & \text{if } 0 = v_{i+1} \neq u_{i+1} = 1 \\ * & \text{otherwise} \end{cases}.$$

Here, $*$ is an unspecified point in $[0,i]$, which serves as a placeholder. We have maps $\mathcal{I}_{u,v}$, given by the obvious identifications

$$\mathcal{M}_{|u|-|v|}(\underline{1},\underline{0}) = \bigtimes_{i=1}^{|u|-|v|-1} [0,i] \to \bigtimes_{i=1}^{n} \mathcal{N}_i(u,v) = \mathcal{M}_{n+1}(u,v).$$

This means that we can consider $\mathfrak{C}(v^{(r)}, \ldots, v^{(1)})$ as a subspace of the cuboid

$$\mathcal{M}_{n+1}(\underline{1},\underline{0}) = \bigtimes_{i=1}^{n} [0,i]$$

and the arrows in the diagram $\mathfrak{C}$ are just inclusions.

In particular for $v \in \{0,1\}^n \setminus \underline{1}$ with $|v| = n-1$ and $v_m = 0$, we have

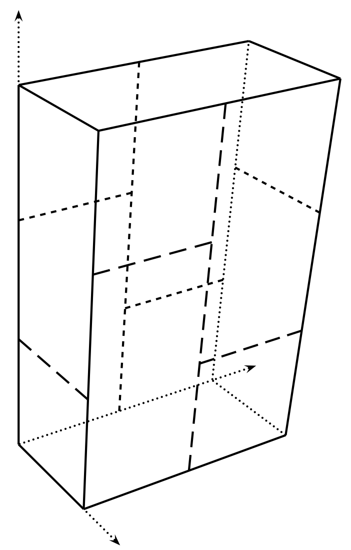
$$B_C(v) = \coprod_{v=\mathfrak{v}(v')} \mathfrak{C}(v') = \coprod_{|v'|=m} \bigtimes_{j=1}^{n} \mathcal{N}_j(v') = \partial_m \mathcal{M}_{n+1}(\underline{1}, \underline{0}),$$

where

$$\mathcal{N}_j(v) = \begin{cases} [|v^j|, j] & \text{if } v_{j+1} = 0 \\ [0, |v^j|] & \text{if } v_{j+1} = 1 \end{cases}.$$

$\partial_m \mathcal{M}_{n+1}(\underline{1}, \underline{0})$ is indeed a disjoint union: If $|v| = m = |u|$ for two $u, v \in \{0,1\}^n \setminus \underline{1}$, consider $j = \max\{i | u_i \neq v_i\}$. Say $1 = u_j \neq v_j = 0$. Then $|v^{j-1}| = |u^{j-1}| + 1$, so $\mathcal{N}_j(u) \cap \mathcal{N}_j(v) = \emptyset$. It is also clear that every point on the boundary $\partial \mathcal{M}_{n+1}(\underline{1}, \underline{0})$ of the cuboid lies in one of these sets $\partial_m \mathcal{M}_{n+1}(\underline{1}, \underline{0})$. So the boundary structure of $\mathcal{M}_{n+1}(\underline{1}, \underline{0})$ is well-defined. ■

The following picture shows $\mathcal{M}_4(\underline{1}, \underline{0})$ as constructed in the proof of the lemma above. Compare this to Figure 7.



The following table lists the vertices of the diagram $\mathfrak{C}$ for chains $(\underline{0} < v < \underline{1})$. For example, the vertex of the chain $(\underline{0} < (0,0,0,1) < \underline{1})$ is the bottom face of the cuboid above.

$\partial_1 \mathcal{M}_4(\underline{1}, \underline{0})$	$\partial_2 \mathcal{M}_4(\underline{1}, \underline{0})$	$\partial_3 \mathcal{M}_4(\underline{1}, \underline{0})$
0001: $[0,1] \times [0,2] \times \{0\}$	0011: $[0,1] \times \{0\} \times [0,1]$	0111: $\{0\} \times [0,1] \times [0,2]$
0010: $[0,1] \times \{0\} \times [1,3]$	0101: $\{0\} \times [1,2] \times [0,1]$	1011: $\{1\} \times [0,1] \times [0,2]$
0100: $\{0\} \times [1,2] \times [1,3]$	1001: $\{1\} \times [1,2] \times [0,1]$	1101: $[0,1] \times \{2\} \times [0,2]$
1000: $\{1\} \times [1,2] \times [1,3]$	0110: $\{0\} \times [0,1] \times [2,3]$	1110: $[0,1] \times [0,2] \times \{3\}$
	1010: $\{1\} \times [0,1] \times [2,3]$	
	1100: $[0,1] \times \{2\} \times [2,3]$	

2.3 The Khovanov Flow Category $\mathscr{C}_K$

In this section, we construct the Khovanov flow category $\mathscr{C}_K = \mathscr{C}_K(L)$ for an oriented link diagram L. We also construct a functor $\mathscr{F} : \mathscr{C}_K \to \mathscr{C}_C(n)$ which is a cover of the cube flow category $\mathscr{C}_C(n)$ in the sense of the next definition.

Definition. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two flow categories. A grading preserving functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is a **cover** of $\mathscr{C}_2$ if for any two objects $\mathbf{x}$ and $\mathbf{y}$ in $\mathrm{ob}\mathscr{C}_1$, the map

$$F_{\mathbf{x},\mathbf{y}} : \mathcal{M}_{\mathscr{C}_1}(\mathbf{x},\mathbf{y}) \to \mathcal{M}_{\mathscr{C}_2}(F(\mathbf{x}), F(\mathbf{y}))$$

is a $(\mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\mathbf{y}) - 1)$-map, a local diffeomorphism and a covering. The cover is **trivial** if the maps $F_{\mathbf{x},\mathbf{y}}$ are trivial coverings for all $\mathbf{x}$ and $\mathbf{y}$.

Construction Plan. The objects of the Khovanov flow category $\mathscr{C}_K$ of a link diagram L are the labelled resolution configurations $(D_L(u), x)$. The grading on objects is the homological grading gr_h from the definition of Khovanov homology (Section 1.3, page 8). The quantum grading gr_q is an additional grading on objects. The cover $\mathscr{F} : \mathscr{C}_K \to \mathscr{C}_C(n)$ sends the objects $(D_L(u), x)$ to their corresponding "vertices" u in the cube flow category. We can make this functor grading-preserving by shifting the gradings of the cube flow category by n_-.
The question of how to define the moduli spaces is predictably more difficult. First of all, we want to separate objects of different quantum gradings. Therefore, we set

$$\mathcal{M}((D_L(u), x), (D_L(v), y)) := \emptyset \quad \text{if} \quad (D_L(v), y) \not\prec (D_L(u), x). \tag{4}$$

Then, for each pair

$$\mathbf{y} = (D_L(v), y) \prec (D_L(u), x) = \mathbf{x}$$

of labelled resolution configurations, we need to find moduli spaces $\mathcal{M}_K(\mathbf{x},\mathbf{y})$ and corresponding maps $\mathscr{F}_{\mathbf{x},\mathbf{y}}$ with the required properties. Since $\mathcal{M}_{\mathscr{C}_C(n)}(u, v)$ is identified with $\mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1}, \underline{0})$ as $\langle |u| - |v| - 1\rangle$-manifolds via

$$\mathcal{I}_{u,v} : \mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1}, \underline{0}) \to \mathcal{M}_{\mathscr{C}_C(n)}(u, v),$$

it is reasonable to suggest that $\mathcal{M}_K(\mathbf{x},\mathbf{y})$ should only depend on $D_L(v) \setminus D_L(u)$ and its decoration $x|$ and $y|$ induced by x and y. By construction, $D_L(v) \setminus D_L(u)$ is a basic index $|u| - |v|$ resolution configuration. Thus, it is enough to construct moduli spaces $\mathcal{M}(D, x, y)$ for basic resolution configurations (D, x, y) and corresponding maps

$$\mathcal{F}_{(D,x,y)} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\mathscr{C}_C(\mathrm{ind}\, D)}(\underline{1}, \underline{0}).$$

Lemma 1/Definition (Khovanov flow category)**.** Suppose that for all basic decorated resolution configurations (D, x, y), we have moduli spaces $\mathcal{M}(D, x, y)$ and maps $\mathcal{F}_{(D,x,y)}$ satisfying the properties listed in the summary on page 29. (In the proof below, we will see where these conditions come from and that they are in fact quite natural.) Setting

$$\mathcal{M}_K(\mathbf{x},\mathbf{y}) := \mathcal{M}(D_L(v) \setminus D_L(u), x|, y|) \quad \text{and} \quad \mathscr{F}_{\mathbf{x},\mathbf{y}} := \mathcal{I}_{u,v}\mathcal{F}_{(D,x,y)}$$

then completes the above construction of flow categories $\mathscr{C}_K(L)$ and covers $\mathscr{F}$ for all oriented link diagrams L. Furthermore, if $\mathscr{C}_K^j(L)$ denotes the full subcategory of $\mathscr{C}_K(L)$ whose objects are exactly those of quantum grading j, then $\mathscr{C}_K(L)$ can be written as the disjoint union of the $\mathscr{C}_K^j(L)$:

$$\mathscr{C}_K(L) = \coprod_j \mathscr{C}_K^j(L).$$

Proof. The final claim follows from condition (4) above. We want to see where the conditions in the summary on page 29 come from and why they ensure that we get the desired properties of $\mathcal{M}_K$ and $\mathscr{F}$.

If

$$\mathbf{y} = (D_L(v), y) \prec \mathbf{z} = (D_L(w), z) \prec \mathbf{x} = (D_L(u), x),$$

we need a composition map

$$\circ : \mathcal{M}(D_L(v) \setminus D_L(w), z|, y|) \times \mathcal{M}(D_L(w) \setminus D_L(u), x|, z|) \to \mathcal{M}(D_L(v) \setminus D_L(u), x|, y|).$$

The map $\circ$ needs to be an embedding into $\partial_m \mathcal{M}(D_L(v) \setminus D_L(u), x|, y|)$ for $m = |w| - |v|$. Also $\circ^{-1}(\partial_i \mathcal{M}(D_L(v) \setminus D_L(u), x|, y|))$ needs to be a particular face of the product on the left-hand side, but this follows from the properties of the $\mathcal{F}$'s, as we will see below.
Let

$$(D, x', y') := (D_L(v) \setminus D_L(u), x|, y|) \quad \text{and} \quad (E, z') := (s_{A'}(D), z|),$$

where $A' \subseteq A(D_L(v))$ is such that $D_L(w) = s_{A'}(D_L(v))$. Note that

$$D_L(v) \setminus D_L(w) = D \setminus E, \quad D_L(w) \setminus D_L(u) = E \setminus s(D) \quad \text{and} \quad m = \mathrm{ind}(D \setminus E).$$

Thus, the composition map has indeed the following form:

$$\circ : \mathcal{M}(D \setminus E, z|, y|) \times \mathcal{M}(E \setminus s(D), x|, z|) \to \mathcal{M}(D, x, y),$$

Figure 8 illustrates these definions. Then (FC-3) clearly translates into condition (KF-2).

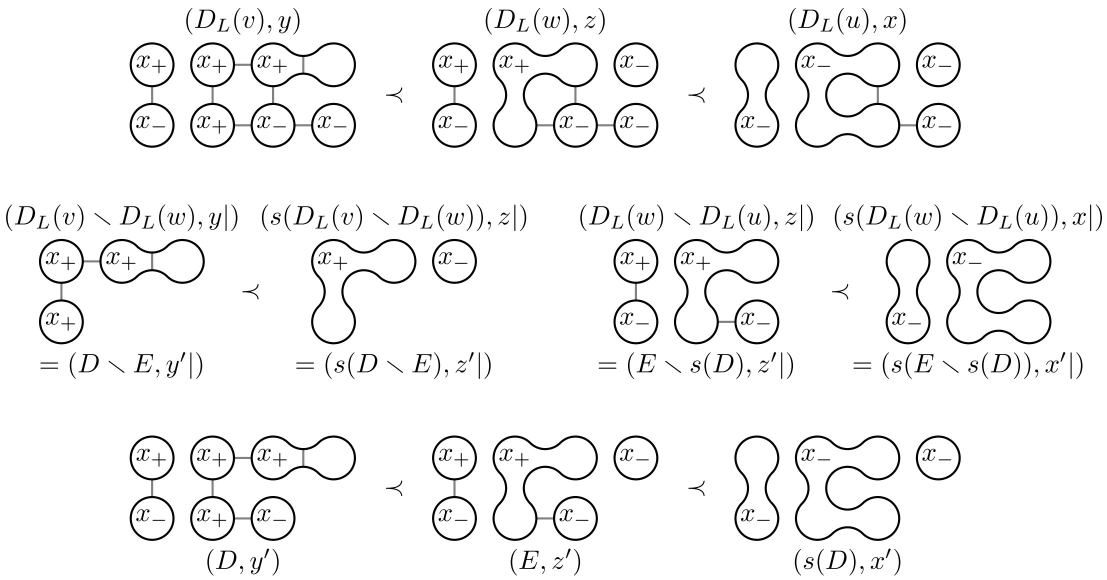


Figure 8: Compositions in the Khovanov flow category

For $\mathscr{F}$ to be a functor, we need to make sure that it is compatible with compositions, i. e. that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{M}_K(\mathbf{z},\mathbf{y}) \times \mathcal{M}_K(\mathbf{x},\mathbf{z}) & \xrightarrow{\circ} & \mathcal{M}_K(\mathbf{x},\mathbf{y}) \\
\big\downarrow \mathscr{F}_{\mathbf{z},\mathbf{y}} \times \mathscr{F}_{\mathbf{x},\mathbf{z}} & & \big\downarrow \mathscr{F}_{\mathbf{x},\mathbf{y}} \\
\mathcal{M}_{\mathscr{C}_C(n)}(w,v) \times \mathcal{M}_{\mathscr{C}_C(n)}(u,w) & \xrightarrow{\circ} & \mathcal{M}_{\mathscr{C}_C(n)}(u,v)
\end{array}$$

Recall the construction of the composition maps in the previous section on page 21. Then this is equivalent to asking for the outer maps in the following diagram to commute:

$$\begin{array}{ccc}
\mathcal{M}_K(\mathbf{z},\mathbf{y}) \times \mathcal{M}_K(\mathbf{x},\mathbf{z}) & \xrightarrow{\circ} & \mathcal{M}_K(\mathbf{x},\mathbf{y}) \\
\mathscr{F}_{\mathbf{z},\mathbf{y}} \times \mathscr{F}_{\mathbf{x},\mathbf{z}} \big\downarrow & & \big\downarrow \mathscr{F}_{\mathbf{x},\mathbf{y}} \\
\mathcal{M}_{\mathscr{C}_C(n)}(w,v) \times \mathcal{M}_{\mathscr{C}_C(n)}(u,w) & \xrightarrow{\circ} & \mathcal{M}_{\mathscr{C}_C(n)}(u,v) \\
\mathcal{I}_{w,v} \times \mathcal{I}_{u,w} \big\uparrow & & \big\uparrow \mathcal{I}_{u,v} \\
\mathcal{M}_{\mathscr{C}_C(|w|-|v|)}(\underline{1},\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(|u|-|w|)}(\underline{1},\underline{0}) & \longrightarrow & \mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1},\underline{0}) \\
\mathcal{I}_{w',\underline{0}} \times \mathcal{I}_{\underline{1},w'} \big\downarrow & \nearrow^{\circ} & \\
\mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(w',\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1},w') & &
\end{array}$$

Here, w' is the element in $\{0,1\}^{|u|-|v|}$ obtained by omitting those entries of w which coincide in u and v. If we now replace the $\mathscr{F}_.$ and $\mathcal{I}_.$ by the $\mathcal{F}_.$, we finally get:

$$\begin{array}{ccc}
\mathcal{M}_K(\mathbf{z},\mathbf{y}) \times \mathcal{M}_K(\mathbf{x},\mathbf{z}) & \xrightarrow{\circ} & \mathcal{M}_K(\mathbf{x},\mathbf{y}) \\
\big\downarrow \mathcal{F}_{(D_L(v)\setminus D_L(u),z|,y|)} \times \mathcal{F}_{(D_L(w)\setminus D_L(u),x|,z|)} & & \\
\mathcal{M}_{\mathscr{C}_C(|w|-|v|)}(\underline{1},\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(|u|-|w|)}(\underline{1},\underline{0}) & & \Big\downarrow \mathcal{F}_{(D_L(v)\setminus D_L(u),x,y)} \\
\big\downarrow \mathcal{I}_{w',\underline{0}} \times \mathcal{I}_{\underline{1},w'} & & \\
\mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(w',\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1},w') & \xrightarrow{\circ} & \mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1},\underline{0})
\end{array}$$

This is the commutative diagram condition in (KF-1).

Every $\mathscr{F}_{\mathbf{x},\mathbf{y}}$ and hence every $\mathcal{F}_{(D,x,y)}$ needs to be a cover and a local diffeomorphism, which is condition (KF-3). Since the moduli spaces of the cube flow category are contractible (being homotopic to disks), the cover is automatically trivial. This also implies condition (FC-1).

Finally, for $\mathscr{F}_{\mathbf{x},\mathbf{y}}$ to be $(n-1)$-maps, the maps $\mathcal{F}_{(D,x,y)}$ should also be $(n-1)$-maps. Then, the above commutative diagram implies the last condition in (FC-2) on $\circ$. In fact,

$$\begin{aligned}
&\circ^{-1}\left(\partial_i\mathcal{M}(D,x,y)\right) = \circ^{-1}(\mathcal{F}^{-1}_{(D,x,y)}(\partial_i\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1},\underline{0}))) \\
&= (\mathcal{F}_. \times \mathcal{F}_.)^{-1}(\mathcal{I}_. \times \mathcal{I}_.)^{-1} \circ^{-1} \left(\partial_i\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1},\underline{0})\right) \\
&= (\mathcal{F}_. \times \mathcal{F}_.)^{-1}(\mathcal{I}_. \times \mathcal{I}_.)^{-1} \left(\begin{cases} \partial_i\mathcal{M}_{\mathscr{C}_C(n)}(w',\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(n)}(\underline{1},w') & \text{for } i < m \\ \mathcal{M}_{\mathscr{C}_C(n)}(w',\underline{0}) \times \partial_{m-i}\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1},w') & \text{for } i > m \end{cases} \right)
\end{aligned}$$

Now use the fact that the $\mathcal{F}_.$ and $\mathcal{I}_.$ respect the $\langle n\rangle$-manifold structures. ■

Summary. The moduli spaces $\mathcal{M}(D,x,y)$ and maps $\mathcal{F}_{(D,x,y)}$ for basic index n decorated resolution configurations (D,x,y) need to satisfy the following properties:

(KF-1) If $(E,z) \in P(D,x,y)$ and $(D,y) \prec_m (E,z)$, then

$$\circ : \mathcal{M}(D \smallsetminus E, z|, y|) \times \mathcal{M}(E \smallsetminus s(D), x|, z|) \to \mathcal{M}(D,x,y)$$

is an embedding into $\partial_m \mathcal{M}(D,x,y)$ and it satisfies the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(D \smallsetminus E, z|, y|) \times \mathcal{M}(E \smallsetminus s(D), x|, z|) & \xrightarrow{\circ} & \mathcal{M}(D,x,y) \\
\downarrow \mathcal{F}_{(D \smallsetminus E, z|, y|)} \times \mathcal{F}_{(E \smallsetminus s(D), x|, z|)} & & \\
\mathcal{M}_{\mathscr{C}_C(m)}(\underline{1},\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(n-m)}(\underline{1},\underline{0}) & & \downarrow \mathcal{F}_{(D,x,y)} \\
\downarrow \mathcal{I}_{w,\underline{0}} \times \mathcal{I}_{\underline{1},w} & & \\
\mathcal{M}_{\mathscr{C}_C(n)}(w,\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(n)}(\underline{1},w) & \xrightarrow{\circ} & \mathcal{M}_{\mathscr{C}_C(n)}(\underline{1},\underline{0})
\end{array}$$

where w corresponds to E, i. e. $w_i = 0$ iff the i^{th} arc in D is an arc in E.

(KF-2) For all $1 \leq i \leq n-1$,

$$\partial_i \mathcal{M}(D,x,y) = \coprod_{(D,x) \prec_m (E,z) \prec (s(D),y)} \circ(\mathcal{M}(D \smallsetminus E, z|, y|) \times \mathcal{M}(E \smallsetminus s(D), x|, z|)).$$

(KF-3) $\mathcal{F}_{(D,x,y)}$ is an $(n-1)$-map, a cover of $\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1},\underline{0})$ and a local diffeomorphism. In fact, $\mathcal{F}_{(D,x,y)}$ is a trivial cover.

We will now inductively construct $\mathcal{M}(D,x,y)$ and $\mathcal{F}_{(D,x,y)}$ satisfying these properties. We do the induction step first.

Induction Step. Suppose we have already constructed moduli spaces $\mathcal{M}(D,x,y)$ and maps $\mathcal{F}_{(D,x,y)}$ for all basic index $\leq n$ resolution configurations (D,x,y) satisfying conditions (KF-1)–(KF-3). Then, analogous to the construction of the cube flow category, define the diagram $\mathfrak{D}$ consisting of vertices of the form

$$\mathfrak{D}((E_r,z_r),\dots,(E_1,z_1)) := \mathcal{M}(D \smallsetminus E, z_1|, y|) \times \cdots \times \mathcal{M}(E_r \smallsetminus s(D), x|, z_r|),$$

where $(D,y) \prec (E_1,z_1) \prec \cdots \prec (E_r,z_r) \prec (s(D),x)$ for some $r \geq 1$, and arrows being the composition maps

$$\begin{array}{c}
\mathcal{M}(\dots) \times \cdots \times \mathcal{M}(E_{i-1} \smallsetminus E_i, z_i|, z_{i-1}|) \times \mathcal{M}(E_i \smallsetminus E_{i+1}, z_{i+1}|, z_i|) \times \cdots \times \mathcal{M}(\dots) \\
\downarrow \circ \\
\mathcal{M}(D \smallsetminus E_1, z_1|, y|) \times \cdots \times \mathcal{M}(E_{i-1} \smallsetminus E_{i+1}, z_{i+1}|, z_{i-1}|) \times \cdots \times \mathcal{M}(E_r \smallsetminus s(D), x|, z_r|)
\end{array}$$

for some i. We make $\mathfrak{D}$ into an n-boundary, by taking the disjoint union over certain vertices of $\mathfrak{D}$ and the induced arrows: For every $v \in \{0,1\}^n \smallsetminus \underline{1}$, we define

$$B_K(v) := \coprod \mathfrak{D}((E_r,z_r),\dots,(E_1,z_1)), \tag{5}$$

where $r = n - |v|$ and the coproduct is over all sequences

$$(D, y) \prec (E_1, z_1) \prec \cdots \prec (E_r, z_r) \prec (s(D), x)$$

such that

$$v_i = 0 \Leftrightarrow \exists j : (D, y) \prec_i (E_j, z_j).$$

In particular, if $|v| = n - 1$ and say $v_m = 0$, then

$$B_K(v) = \coprod_{(D,x) \prec_m (E,z) \prec (s(D),y)} \mathcal{M}(D \setminus E, z|, y|) \times \mathcal{M}(E \setminus s(D), x|, z|).$$

This will be $\partial_m \mathcal{M}(D, x, y)$. We compare this to the very similar description of the cube flow category in the previous section, starting on page 21. We have maps

$$\mathcal{IF} \times \cdots \times \mathcal{IF} : \mathfrak{D}((E_r, z_r), \ldots, (E_1, z_1)) \to \mathfrak{C}(v^{(r)}, \ldots, v^{(1)}),$$

where the $v^{(i)}$ correspond to the E_i, i. e. $v_j^{(i)} = 0$ iff the j^{th} arc in D is an arc in E_i. By condition (KF-1), these maps form a map of diagrams $\mathfrak{D} \to \mathfrak{C}$ which induces a natural transformation $B_K \to B_C$. By induction hypothesis, we can apply the lemma of Section 2.1 on page 17. Hence, the induced map $\operatorname{colim} \mathcal{F} : \operatorname{Colim} B_K \to \operatorname{Colim} B_C$ is a covering map. The rest of the induction step is taken care of by the following lemma.

Lemma 2. There is a space $\mathcal{M}(D, x, y)$ together with a map

$$\mathcal{F}_{(D,x,y)} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0}),$$

such that the conditions (KF-1)–(KF-3) are satisfied iff $\operatorname{colim} \mathcal{F}$ is a trivial cover. In particular, this hypothesis is true for $n \geq 3$. For $n \geq 2$, $\mathcal{M}(D, x, y)$ and the map $\mathcal{F}_{(D,x,y)}$ are uniquely determined by (KF-1)–(KF-3) up to smooth deformation (fixing the boundary).

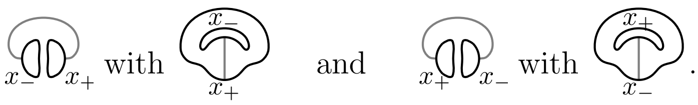
Proof. If $\mathcal{F}_{(D,x,y)}$ is trivial, then the restriction to $\operatorname{colim} \mathcal{F}$ is also trivial. Conversely, if $\operatorname{colim} \mathcal{F}$ is trivial, then $\partial \mathcal{M}(D, x, y)$ consists of a (finite) number of copies of $\partial \mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$, so we can define $\mathcal{M}(D, x, y)$ correspondingly as a product of copies of $\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$. The conditions (KF-1)–(KF-3) are satisfied by construction.
Recall that $\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$ is an n-dimensional disk and $\partial \mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$ an $(n-1)$-dimensional sphere. All coverings of spheres are trivial if the dimension of the spheres is ≥ 2. Hence, the hypothesis is true for $n \geq 3$.
Uniqueness follows from the fact that $\partial \mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$ is connected for $n \geq 2$. ■

Therefore, for the start of induction, we only need to consider basic index 1 and 2 decorated resolution configurations and check whether $\mathcal{F}_{(D,x,y)}$ is indeed a trivial covering for index equal to 3.

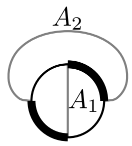
Basic Index 1 Decorated Resolution Configurations. Unsurprisingly, we define $\mathcal{M}(D, x, y)$ to be a single point if (D, x, y) is an index 1 basic resolution configuration. Since $\mathcal{M}_{\mathscr{C}_C(1)}(\underline{1}, \underline{0})$ also consists of a single point, we have no choice for $\mathcal{F}_{(D,x,y)}$ and the conditions (KF-1)–(KF-3) are vacuously satisfied.

Basic Index 2 Decorated Resolution Configurations. As we saw in the proof in Section 1.3 of $\delta^2 = 0$ in the Khovanov chain complex, there are essentially two cases. There are either two labelled resolution configurations between (D, y) and $(s(D), x)$ or four, in which case D is a ladybug configuration. Also $\mathcal{M}_{\mathscr{C}_C(2)}(\underline{1}, \underline{0})$ is an interval, so in the first case, $\mathcal{M}(D, x, y)$ consists of one interval and in the second of two intervals, since by (KF-3), we want $\mathcal{F}_{(D,x,y)}$ to be a trivial cover.
Thus, in the first case, $\mathcal{M}(D, x, y)$ and $\mathcal{F}_{(D,x,y)}$ are uniquely determined (up to smooth deformation). However, in the case of the ladybug configuration, we have two choices that are equally natural: Consider Figure 2 in Section 1.3 again. $\mathcal{F}$ sends the two labelled resolution configurations at each vertex in the middle to the same vertex in the cube flow category. Hence, the endpoints of each interval belong to different vertices, but there are two ways to match them, namely either

or



We choose the first one, but we have to make precise what distinguishes these two choices: Consider the four components in $D \smallsetminus (A_1 \cup A_2)$. Then there is one pair of components each of which lies to the right of the boundary points of the arcs A_1 and A_2 if one travels along the arcs towards their boundary points. It is highlighted in the following picture of the ladybug configuration:



Then we choose the matching where the labels of the circles containing the components of this pair agree. So once we have specified an orientation on S^2, we can tell which of the two possible matchings to choose. We call the one above the **ladybug matching**.

Basic Index 3 Decorated Resolution Configurations. If we go back to Lemma 2 above, we just need to show that the covering map

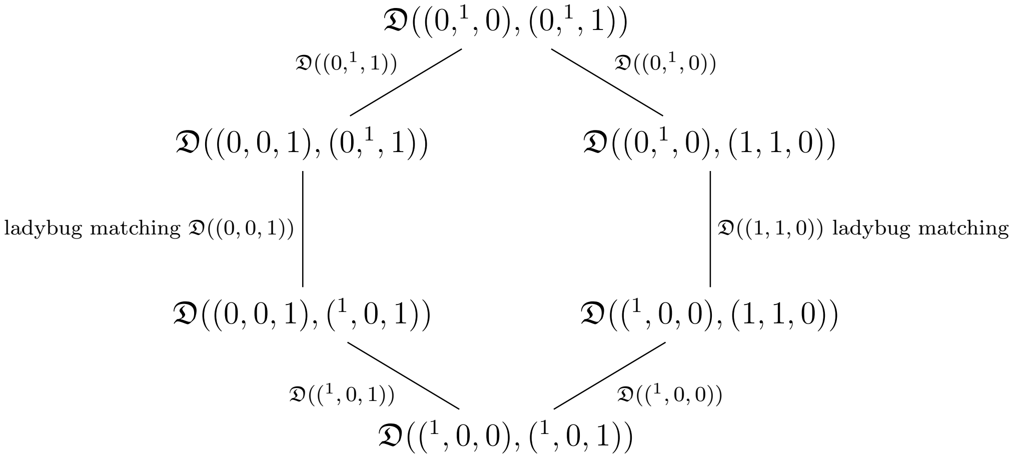
$$\operatorname{colim} \mathcal{F} : \partial\mathcal{M}(D, x, y) \to \partial\mathcal{M}_{\mathscr{C}_C(3)}(\underline{1}, \underline{0})$$

is trivial. Similarly to the proof of $\delta^2 = 0$ in Section 1.3, this is essentially done by case analysis. Since $\operatorname{colim} \mathcal{F}$ is a covering map, it is clear that the number of vertices in each cycle of $\partial\mathcal{M}(D, x, y)$ is divisible by 6. So if $\#P(D, x, y) = 6$, we are already done. This is the case, if we can apply the Splitting Lemma or the n-arc Lemma from Section 1.4 to see $P(D, x, y) \cong \{0, 1\}^3$. There are essentially two other cases:

1) If we can apply the Splitting Lemma only once and do not obtain the situation of the n-arc Lemma, then by the proof of the Index 2 Lemma in Section 1.3, D' in the splitting

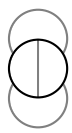
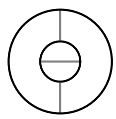
$$P(D,x,y) \cong P(D',x',y') \times \{0,1\}$$

is the ladybug configuration. We introduce the following notation for the vertices in $P(D',x',y')$: The vertices of one pair in the ladybug matching are denoted by $({}^1,0)$ and $(0,{}^1)$ and correspondingly those of the other pair by $({}_1,0)$ and $(0,{}_1)$. If we now recall the glueing construction which builds $\operatorname{colim}\mathcal{F}$ from the data of the diagram $\mathfrak{D}$, it is clear that $\operatorname{colim}\mathcal{F}$ consists of two 6-cycles, one of which is the following



We obtain the second 6-cycle if we replace ^{1} by ${}_1$.

2) If we cannot apply the Splitting Lemma at all and we are not in the situations of the n-arc Lemma, then it is not hard to see that we are in one of the following two situations, which are dual to each other:

 and  .

Lipshitz and Sarkar show in [LS11, pp. 41–43] that $\partial\mathcal{M}(D,x,y)$ is a disjoint union of 6-cycles iff this is true for $\partial\mathcal{M}(D^*,y^*,x^*)$ and therefore, they only check the first resolution configuration above. We will look at the other case and claim that the first one is very similar, referring the reader to [LS11] for reassurance.
The only labellings admitting a decoration of these two resolution configurations are those with x_+ on all circles. The full diagram for $P(D,\{x_-\},\{x_+,x_+\})$ is drawn in Figure 9. The raising and lowering of the 1's in the notation $(\cdot,\cdot,\cdot)$ for the labelled resolution configurations in $P(D,\{x_-\},\{x_+,x_+\})$ is such that for every vertex where two valid labellings exist, ^{1} denotes the resolution configuration with labelling x_+ of a circle with $x_\pm$ and with labelling x_- of a circle with $x_\mp$; similarly for ${}_1$. The reason, why $\partial\mathcal{M}(D,x,y)$ is indeed a disjoint union of 6-cycles, is that this labelling is consistent with the ladybug matchings of the two ladybug configurations which occur. As can be easily seen, raised 1's are matched with raised 1's and the same for lowered 1's.

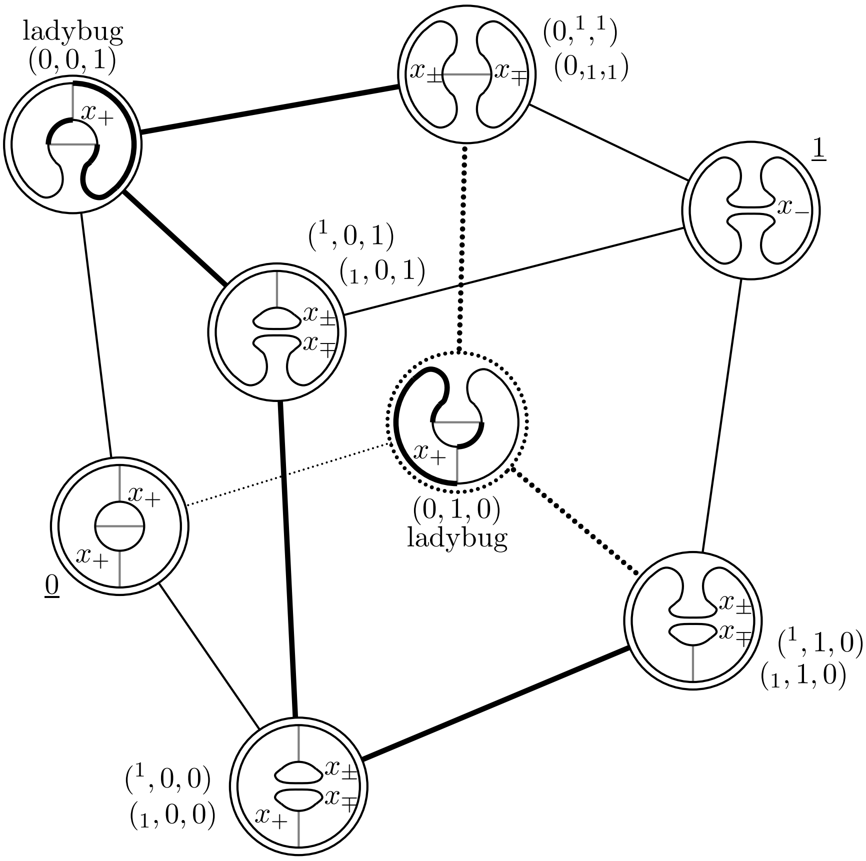
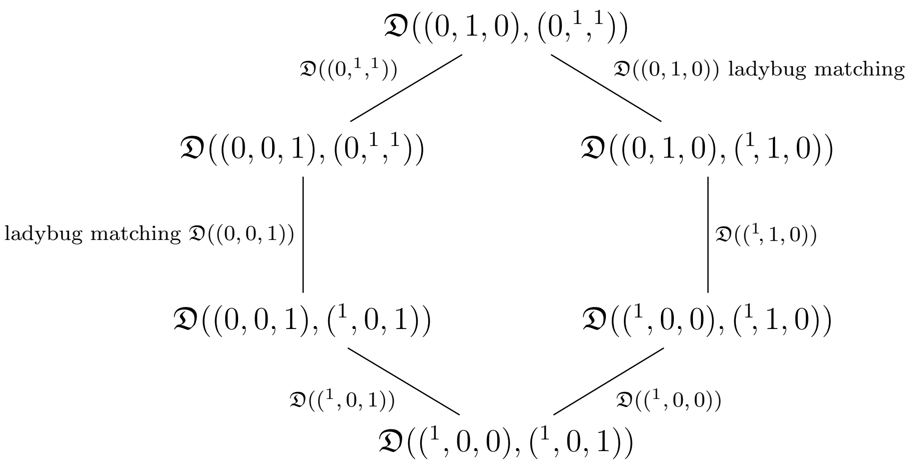


Figure 9: A diagram for the poset in the second case. A good way to read it is to start at the vertex (1, 0, 0) and to verify the matching on the "front" face (1,0,0)–(1,1,1). Then consider the ladybug matchings.

In fact, we get the following 6-cycle:



The other 6-cycle is obtained by replacing ^{1} by $_1$.

2.4 From Framed Flow Categories to CW Complexes

Definitions. Given an $(n+1)$-tuple $\mathbf{d} = (d_0, \dots, d_n)$, define an $\langle n \rangle$-manifold $\mathbb{E}^{\mathbf{d}}$ by setting

$$\begin{aligned} \mathbb{E}^{\mathbf{d}} &:= \mathbb{R}^{d_0} \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^{d_{i-1}} \times \mathbb{R}^+ \times \mathbb{R}^{d_i} \times \cdots \times \mathbb{R}^+ \times \mathbb{R}^{d_n} \quad \text{and} \\ \partial_i \mathbb{E}^{\mathbf{d}} &:= \mathbb{R}^{d_0} \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^{d_{i-1}} \times \{0\} \times \mathbb{R}^{d_i} \times \cdots \times \mathbb{R}^+ \times \mathbb{R}^{d_n} \quad \text{for } 1 \le i \le n. \end{aligned}$$

Given some $\langle n \rangle$-manifold X, define a **neat immersion/neat embedding** of X relative to $\mathbf{d}$ to be an immersion/embedding $X \to \mathbb{E}^{\mathbf{d}}$ which is also an n-map such that for all $v, u \in \{0,1\}^n$ with $u < v$, $X(v) \perp \mathbb{E}^{\mathbf{d}}(u)$.
More generally, given a sequence $\mathbf{d} = (d_i)_{i \in \mathbb{Z}}$ and some $a, b \in \mathbb{Z}$ with $a > b$, define

$$\mathbb{E}^{\mathbf{d}}[a:b] := \mathbb{E}^{(d_b, \dots, d_{a-1})}.$$

Then, given some flow category $\mathscr{C}$, define a **neat immersion/embedding of $\mathscr{C}$ relative to d** to be a set of neat immersions/neat embeddings

$$\iota_{i,j} : \mathcal{M}(i,j) := \coprod_{\substack{\mathrm{gr}(x)=i \\ \mathrm{gr}(y)=j}} \mathcal{M}(x,y) \to \mathbb{E}^{\mathbf{d}}[i:j]$$

satisfying the commutative diagrams

$$\begin{array}{ccc} \mathcal{M}(k,j) \times \mathcal{M}(i,k) & \hookrightarrow & \mathcal{M}(i,j) \\ \downarrow{\scriptstyle \iota_{k,j} \times \iota_{i,k}} & & \downarrow{\scriptstyle \iota_{i,j}} \\ \mathbb{E}^{\mathbf{d}}[k:j] \times \mathbb{E}^{\mathbf{d}}[i:k] & \hookrightarrow & \mathbb{E}^{\mathbf{d}}[i:j] \end{array}$$

Lemma 1. Every flow category $\mathscr{C}$ admits a neat embedding relative to some $\mathbf{d}$.

Proof. See [LS11, Lemma 3.16]. ∎

Definitions. Given a neat embedding ι of a flow category $\mathscr{C}$, we can consider the normal bundles $\nu_{\iota_{\mathbf{x},\mathbf{y}}}$ of the embeddings $\iota_{\mathbf{x},\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y} \in \mathrm{ob}\mathscr{C}$. A **framing** of $\nu_{\iota_{\mathbf{x},\mathbf{y}}}$ is a specification of global sections of $\nu_{\iota_{\mathbf{x},\mathbf{y}}}$ which form a basis of each fibre. A **coherent framing** of a flow category $\mathscr{C}$ is a framing of the normal bundles which is consistent with the flow category structure, i. e. the product framing of $\nu_{\iota_{\mathbf{z},\mathbf{y}}} \times \nu_{\iota_{\mathbf{x},\mathbf{z}}}$ coincides with the induced framing of the pullback $\circ^* \nu_{\iota_{\mathbf{x},\mathbf{y}}}$ for all $\mathbf{x}, \mathbf{y} \in \mathrm{ob}\mathscr{C}$. Such a flow category is called a **framed flow category**.

Let $(\mathscr{C}, \iota, \varphi)$ be a framed flow category, where φ is a framing of the normal bundle of a neat embedding ι of $\mathscr{C}$ relative to some $\mathbf{d}$. We are going to construct a chain complex $C^\bullet(\mathscr{C})$ and a CW complex $|\mathscr{C}|$ associated with the given framed flow category, such that the cellular chain complex of $|\mathscr{C}|$ is equal to $C^\bullet(\mathscr{C})$ up to some grading shift. The framing of the flow category is needed to get the correct signs/orientations in these two constructions. Let us first look at the construction of $C^\bullet(\mathscr{C})$.

Construction of $C^\bullet(\mathscr{C})$. The i^{th} chain module $C^i(\mathscr{C})$ of $C^\bullet(\mathscr{C})$ is freely generated by the objects of $\mathscr{C}$ of grading i. The differentials d are defined by

$$d(\mathbf{y}) := \sum_{\mathbf{y} \prec_1 \mathbf{x}} |\mathcal{M}(\mathbf{x}, \mathbf{y})| \, \mathbf{x},$$

where $|\mathcal{M}(\mathbf{x}, \mathbf{y})|$ means the following: For $\mathbf{y} \prec_1 \mathbf{x}$, $\mathcal{M}(\mathbf{x}, \mathbf{y})$ is a compact $\langle 0 \rangle$-manifold, i. e. it consists of finitely many points. Furthermore, $\mathcal{M}(\mathbf{x}, \mathbf{y})$ is framed, i. e. there is a neat embedding $\iota_{\mathbf{x},\mathbf{y}}$ of these points into some $\mathbb{R}^n$ and at each point p, there is a basis of the normal bundle of $\iota_{\mathbf{x},\mathbf{y}}$, which is equal to $T_{\iota_{\mathbf{x},\mathbf{y}}(p)}\mathbb{R}^n \cong \mathbb{R}^n$. If the orientation of the point $\iota_{\mathbf{x},\mathbf{y}}(p)$ is positive, i. e. the determinant of the basis vectors is positive, we write $o(p) = +1$ and $o(p) = -1$ otherwise. Then $|\mathcal{M}(\mathbf{x}, \mathbf{y})|$ is defined as the sum of the $o(p)$'s, where p varies over all points in $\mathcal{M}(\mathbf{x}, \mathbf{y})$. We say, this chain complex $C^\bullet(\mathscr{C})$ refines the framed flow category $\mathscr{C}$.

Lemma 2. This chain complex is well-defined, i. e. $d^2 = 0$.

Proof. We have to show that for all $\mathbf{x}, \mathbf{y} \in \text{ob}\mathscr{C}$ with $\mathbf{x} \succ_2 \mathbf{y}$,

$$\sum_{\mathbf{x} \succ_1 \mathbf{z} \succ_1 \mathbf{y}} |\mathcal{M}(\mathbf{z}, \mathbf{y})| \cdot |\mathcal{M}(\mathbf{x}, \mathbf{z})| = 0.$$

Since $\mathcal{M}(\mathbf{x}, \mathbf{y})$ is compact, it is a disjoint union of (finitely many) intervals. Hence, the boundary points come in pairs. Since

$$\partial \mathcal{M}(\mathbf{x}, \mathbf{y}) = \coprod_{\mathbf{x} \succ_1 \mathbf{z} \succ_1 \mathbf{y}} \mathcal{M}(\mathbf{z}, \mathbf{y}) \times \mathcal{M}(\mathbf{x}, \mathbf{z}),$$

it suffices to show that, given an interval in $\mathcal{M}(\mathbf{x}, \mathbf{y})$ together with endpoints $a \times b \in \mathcal{M}(\mathbf{z}, \mathbf{y}) \times \mathcal{M}(\mathbf{x}, \mathbf{z})$ and $a' \times b' \in \mathcal{M}(\mathbf{z}', \mathbf{y}) \times \mathcal{M}(\mathbf{x}, \mathbf{z}')$, $o(a) \cdot o(b) = -o(a') \cdot o(b')$. By the definition of the product framing, this is equivalent to $o(a \times b) = -o(a' \times b')$. But the framing is coherent, i. e. it is induced by the framing of the interval with endpoints $a \times b$ and $a' \times b'$, so the orientation of the endpoint is indeed opposite to each other. (For this, consider for example an orientation on the interval, i. e. a nonvanishing tangent vector field of the embedded interval; it will point in opposite directions on the boundary.) ∎

Construction of $|\mathscr{C}|$. We now turn to the construction of a CW complex. Suppose, $B \leq \text{gr}(\mathbf{x}) \leq A$ for all $\mathbf{x} \in \text{ob}\mathscr{C}$ and some integers A and B. Then $|\mathscr{C}|$ consists of

- one 0-cell e_0 as a basepoint and
- one cell $\mathcal{C}(\mathbf{x})$ of dimension $C + \text{gr}(\mathbf{x})$ for every $\mathbf{x} \in \text{ob}\mathscr{C}$, where

$$C := d_B + \cdots + d_{A-1} - B.$$

Next, extend the embeddings $\iota_{i,j}$ via the normal bundles to

$$\iota^e_{i,j} : \mathcal{M}(i,j) \times [-\varepsilon, \varepsilon]^{d_j + \cdots + d_{i-1}} \to \mathbb{R}^{\mathbf{d}}[i : j]$$

and choose some positive real number R such that for all $i, j \in [B, A]$

$$\text{im}(\iota^e_{i,j}) \subseteq [-R, R]^{d_j} \times [0, R] \times \cdots \times [0, R] \times [-R, R]^{d_{i-1}}.$$

Now, we are ready to define the cells and attaching maps. Suppose we have already constructed the $(C+m-1)$-skeleton of $|\mathscr{C}|$. Take some object $\mathbf{x}$ in grading m. Define

$$\begin{aligned}\mathcal{C}(\mathbf{x}) :=& [0,R]\times[-R,R]^{d_B}\times\ldots\times[-R,R]^{d_{m-1}}\times\{0\}\times[-\varepsilon,\varepsilon]^{d_m}\times\ldots\times\{0\}\times[-\varepsilon,\varepsilon]^{d_{A-1}}\\ \subseteq& \ \mathbb{R}^+\times \mathbb{R}^{d_B}\times\ldots\times\mathbb{R}^{d_{m-1}}\times\mathbb{R}^+\times\mathbb{R}^{d_m}\times\ldots\times\mathbb{R}^+\times\mathbb{R}^{d_{A-1}}.\end{aligned}$$

For objects $\mathbf{y}$ with grading $n<m$, let

$$\mathcal{C}'_{\mathbf{y}}(\mathbf{x}) := \iota^e_{m,n}(\mathcal{M}(\mathbf{x},\mathbf{y})\times[-\varepsilon,\varepsilon]^{d_n+\cdots+d_{m-1}}).$$

Then, define $\mathcal{C}_{\mathbf{y}}(\mathbf{x})$ to be

$$[0,R]\times[-R,R]^{d_B}\times\cdots\times[-R,R]^{d_{n-1}}\times\{0\}\times\mathcal{C}'_{\mathbf{y}}(\mathbf{x})\times\{0\}\times[-\varepsilon,\varepsilon]^{d_m}\times\cdots\times\{0\}\times[-\varepsilon,\varepsilon]^{d_{A-1}}\ ,$$

which is on the boundary of $\mathcal{C}(\mathbf{x})$. Via $\iota^e_{n,m}$, we have an isomorphism between $\mathcal{C}_{\mathbf{y}}(\mathbf{x})$ and

$$\begin{aligned}[0,R]\times[-R,R]^{d_B}\times\cdots\times[-R,R]^{d_{n-1}}\times\{0\}\times\mathcal{M}(\mathbf{x},\mathbf{y})\times[-\varepsilon,\varepsilon]^{d_n+\cdots+d_{m-1}}\\ \times\{0\}\times[-\varepsilon,\varepsilon]^{d_m}\times\cdots\times\{0\}\times[-\varepsilon,\varepsilon]^{d_{A-1}},\end{aligned}$$

which is just $\mathcal{M}(\mathbf{x},\mathbf{y})\times\mathcal{C}(\mathbf{y})$. Then the attaching map on $\mathcal{C}_{\mathbf{y}}(\mathbf{x})$ is defined to be the projection onto $\mathcal{C}(\mathbf{y})$ via this identification. The remaining points in $\partial\mathcal{C}(\mathbf{x})\setminus\mathcal{C}_{\mathbf{y}}(\mathbf{x})$ are mapped to the basepoint.

For an illustration of this construction, see the example on page 41. One can show that $|\mathscr{C}|$ is a well-defined CW complex and that it is constructed in such a way that its cellular chain complex equals the chain complex which $\mathscr{C}$ refines modulo some degree shift. To be precise:

Lemma 3. For any framed flow category $(\mathscr{C},\iota,\phi)$, there is a natural isomorphism between the chain complex $C^\bullet(\mathscr{C})$ refining $\mathscr{C}$ and the reduced cellular cochain complex $\tilde{C}^\bullet(|\mathscr{C}|)$ of $|\mathscr{C}|$ with cohomological grading shifted by $C=d_B+\cdots+d_{A-1}-B$ so that elements of degree C become degree 0 elements:

$$\tilde{C}^\bullet(|\mathscr{C}|)[-C]\cong C^\bullet(\mathscr{C}).$$

Proof. See [LS11, Lemma 3.24]. ■

It is not hard to see that the construction is independent of the choice of the real numbers R and ε [LS11, Lemma 3.25].

Lemma 4. Let $\mathscr{C}$ be the direct product of some flow categories $\mathscr{C}^i$, $i\in\mathbb{Z}$. Then

$$|\mathscr{C}| = \bigvee_j |\mathscr{C}^j|,$$

where we use the the restrictions of the neat embedding and framing of $\mathscr{C}$ for $\mathscr{C}^i$ and the same constants in the constructions of the CW complexes.

Proof. This immediately follows from the construction. ■

Lemma 5. Given a framed flow category $(\mathscr{C}, \iota, \varphi)$, where ι is a neat embedding relative to some $\mathbf{d}$, let $|\mathscr{C}|_{\iota,\varphi,A,B}$ denote the CW complex constructed above. Furthermore, given $\mathbf{d}' \geq \mathbf{d}$, let ι' be the neat embedding of $\mathscr{C}$ into $\mathbb{E}^{\mathbf{d}'}$ induced by $\mathbb{E}^{\mathbf{d}} \hookrightarrow \mathbb{E}^{\mathbf{d}'}$ and φ' be some framing such that the restriction of φ' equals φ. If $A' \geq A$ and $B' \leq B$, then we have a homotopy equivalence

$$|\mathscr{C}|_{\iota,\varphi,A,B} \sim \Sigma^{C_{\mathbf{d}}(B,A)-C_{\mathbf{d}'}(B',A')}|\mathscr{C}|_{\iota',\varphi',A',B'},$$

where $C_{\mathbf{d}}(B,A) = d_B + \cdots + d_{A-1} - B$ and $C_{\mathbf{d}'}(B',A') = d_{B'} + \cdots + d_{A'-1} - B'$ and Σ^k denotes the k-fold reduced suspension (see below).

Proof. See [LS11, Lemma 3.26]. The proof goes by induction, varying A, B and $\mathbf{d}$ separately by 1. ■

Remark. [H01, pp. 8f] Recall that the suspension SX of a CW complex X is obtained from $X \times I$ by collapsing $X \times \{0\}$ and $X \times \{1\}$ to a point, where I is the interval $[0,1]$ as usual. The reduced version ΣX is obtained by taking a 0-cell x_0 of X and collapsing $x_0 \times I$ in SX to a point. Then SX and ΣX are homotopy equivalent, but the reduced version is "nicer" in some sense. For example, the reduced suspension of the wedge product of two CW complexes X and Y with basepoint x_0 is the wedge product of the reduced suspensions of X and Y, i. e. $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$.
By using the Mayer-Vietoris sequence, we see that $\tilde{H}_*(X) = \tilde{H}_{*+1}(\Sigma X)$ and similarly $\tilde{H}^*(X) = \tilde{H}^{*+1}(\Sigma X)$ for any space X. For homotopy groups, we have the following theorem (see for example [H01, p. 360]).

Theorem (Freudenthal)**.** Let X be an n-connected CW complex, i. e. $\pi_k(X) = 0$ for all $k \leq n$. Then the suspension map

$$\begin{array}{rcl} \pi_i(X) & \to & \pi_{i+1}(\Sigma X), \\ [f : S^i \to X] & \mapsto & [\Sigma f : S^{i+1} = \Sigma S^i \to \Sigma X] \end{array}$$

is an isomorphism for $i < 2n+1$ and a surjection for $i = 2n+1$.

Corollary. If X is n-connected, then ΣX is $(n+1)$-connected. Hence the maps

$$\pi_{i+k}(\Sigma^k X) \to \pi_{i+k+1}(\Sigma^{k+1} X) \tag{6}$$

become isomorphisms for k sufficiently large. We denote the resulting **stable homotopy groups** of X by $\pi_i^s(X)$. More formally, we can consider them as the direct limit of the maps in (6):

$$\pi_i^s(X) = \varinjlim \pi_{i+k}(\Sigma^k X).$$

Definition. [H01, p. 454] A **spectrum** K is a sequence (K_i) of CW complexes K_i with basepoints together with basepoint preserving maps $\Sigma K_i \to K_{i+1}$. A **suspension spectrum** is obtained by taking a space X and defining $K_i := \Sigma^i X$. Then the maps $\Sigma K_i \to K_{i+1}$ are the identities. For a suspension spectrum of a CW complex X, we can define its (co)homology as the (co)homology of X.

The homotopy groups of a spectrum are defined via the direct limit construction above, using the compositions of the suspension maps and the homomorphisms induced by the structure maps $\Sigma K_i \to K_{i+1}$. For the suspension spectrum of a space X, these are exactly the stable homotopy groups of X.

Remark. In view of Lemma 5, we would like to "desuspend" the CW complex $|\mathscr{C}|_{\iota,\phi,A,B}$ $C_{\mathbf{d}}(B,A)$ times. We do this formally by taking the suspension spectrum of $|\mathscr{C}|_{\iota,\phi,A,B}$ and shifting the indices by $C_{\mathbf{d}}(B,A)$:

$$K_{i+C_{\mathbf{d}}(B,A)} := \Sigma^i |\mathscr{C}|_{\iota,\phi,A,B}.$$

We can treat such spectra obtained from suspension spectra by index shifts like suspension spectra. It remains to define how to translate the notion of homotopy equivalence into the "stable setting". However, there are several categories whose objects are spectra (see [WikiS] or [A74]). For our purposes, the following definition seems most appropriate.

Definition. A function $f : K \to K'$ between two spectra $K = (K_i)$ and $K' = (K'_i)$ is a sequence of maps $f_i : K_i \to K'_i$ which commute with the structure maps $\Sigma K_i \to K_{i+1}$ and $\Sigma K'_i \to K'_{i+1}$.
We say that two spectra (K_i) and (K'_i) are stably homotopy equivalent, if there is a function $f : K \to K'$ and some N such that f_i is part of a homotopy equivalence between K_i and K'_i for all $i \geq N$.

With this convention, the construction above gives us a spectrum $K = (K_i)$ as defined in the previous remark which we also want to write as

$$K_i = \Sigma^{i-C_{\mathbf{d}}(B,A)} |\mathscr{C}|_{\iota,\phi,A,B},$$

the suspension spectrum of $|\mathscr{C}|_{\iota,\phi,A,B}$, formally desuspended $C_{\mathbf{d}}(B,A)$ times. Lemma 5 then tells us that if $K' = (K'_i)$ is the spectrum defined by

$$K'_i := \Sigma^{i-C_{\mathbf{d}'}(B',A')} |\mathscr{C}|_{\iota',\phi',A',B'},$$

then K and K' are stably homotopy equivalent.

Remark. In [A74, p. 150], Adams defines a stable homotopy equivalence between two (CW) spectra K and K' as a map $f : K \to K'$ such that the induced homomorphisms on the homotopy groups of K and K' are isomorphisms. Here, "map" has a different meaning than the term "function" defined above. Our spectra constructed from a given framed flow category would still be stably homotopy equivalent in this different sense; nevertheless, we prefer to stick to our straightforward, but potentially naïve definition above.

2.5 The Khovanov Spectrum $\mathcal{X}_{Kh}$

We now want to construct a Khovanov homotopy type for an oriented link diagram. For this, we need a coherent framing of some embedding of the Khovanov flow category. Lipshitz and Sarkar do this in the following way.

A Framing for the Khovanov Flow Category. We choose some embedding of the cube flow category and a corresponding coherent framing of it. This coherent framing has the following property: If $u >_1 v$ for $u, v \in \{0,1\}^n$, then the embedded point $\mathcal{M}_{\mathscr{C}_C(n)}(u,v)$ is given a positive orientation iff $s(\mathcal{C}_{u,v})$ is even. For the details of the construction of this framing, we refer the reader to [LS11, Lemma 4.12] and [LS12, Lemma 3.5].
Then, the cover $\mathscr{F} : \mathscr{C}_K \to \mathscr{C}_C(n)$ induces a neat immersion ι relative some $\mathbf{d}$ of the Khovanov flow category with a coherent framing φ. Next, we choose a neat embedding ι' relative $\mathbf{d}'$ of the Khovanov flow category. There are inclusions

$$\mathbb{E}^{\mathbf{d}} \hookrightarrow \mathbb{E}^{\mathbf{d}+\mathbf{d}'} \quad \text{and} \quad \mathbb{E}^{\mathbf{d}'} \hookrightarrow \mathbb{E}^{\mathbf{d}+\mathbf{d}'},$$

so ι and ι' induce immersions $\iota[\mathbf{d}']$ and $\iota'[\mathbf{d}]$ both relative to $\mathbf{d}+\mathbf{d}'$. One can show that, using a 1-parameter family of immersions between $\iota[\mathbf{d}']$ and $\iota'[\mathbf{d}]$, the coherent framing of $\iota[\mathbf{d}']$ induced by φ induces a coherent framing φ' of the neat embedding $\iota'[\mathbf{d}]$. For details, see [LS11, 3.16–3.21]. The framed flow category $(\mathscr{C}_K, \iota'[\mathbf{d}'], \varphi')$ is called a **perturbation** of $(\mathscr{C}_K, \iota, \varphi)$.

Remark. The choices in the construction of perturbations do not affect the result in an essential way: Given two perturbations $(\mathscr{C}, \iota_1, \varphi_1)$ and $(\mathscr{C}, \iota_2, \varphi_2)$ of some flow category $\mathscr{C}$, there is a 1-parameter family of neat immersions between $\iota_1[\mathbf{d}_1]$ and $\iota_2[\mathbf{d}_2]$ for some sufficienly large $\mathbf{d}_1$ and $\mathbf{d}_2$, along with a corresponding 1-parameter family of framings taking the framing induced by φ_1 to the one induced by φ_2 [LS11, Lemma 3.22].

Main Theorem/Definition (Khovanov homotopy type)**.** Given an oriented link diagram L, let

$$\mathscr{C}_K(L) = \coprod_j \mathscr{C}_K^j(L)$$

be the corresponding Khovanov flow category (see Lemma 1, Section 2.3, page 26). Choose a perturbation $(\mathscr{C}_K, \iota_1, \varphi_1)$ of $\mathscr{C}_K(L)$. It gives rise to a CW complex

$$|\mathscr{C}_K(L)| = \bigvee_j |\mathscr{C}_K^j(L)|$$

(see page 36). We formally de-suspend it C times, where C is the constant in the construction of the CW complex $|\mathscr{C}_K(L)|$. We denote the resulting spectrum by $\mathcal{X}_{Kh}(L)$. Since the wedge product is compatible with reduced suspensions, we get

$$\mathcal{X}_{Kh}(L) = \bigvee_j \mathcal{X}_{Kh}^j(L).$$

Then the following is true:

1. The spectra $\mathcal{X}^j_{Kh}(L)$ are independent of all choices in their construction.

2. These spectra are invariants of the oriented link represented by L.

3. Their (reduced) cohomology return the Khovanov homology groups:

$$\tilde{H}^i(\mathcal{X}^j_{Kh}(L)) = Kh^{i,j}(L)$$

Some remarks on the proof. The proof of first claim requires some careful analysis of all the choices involved in the construction. We have already addressed some issues in remarks in this and the previous section. For example, we know that the choice of constants in the construction from the framed flow category to the suspension spectrum has no influence on the stable homotopy type by Lemma 5 of the previous section on page 37. But we would still have to check independence of the choice of the ordering of the crossings, the framing of the cube flow category or the ladybug matching, for example. We refer the reader to [LS11, propositions 6.1 and 6.5] for the details.
For the second claim, we have to study Reidemeister moves. Invariance under the first Reidemeister move is shown in Lemma 2 below. The argument for the second and third Reidemeister moves are more involved, but the essential ideas are the same. We also have to take care of the orientation of the link, but this is not too difficult, since the orientation only affects the gradings. For details, see [LS11, Section 6]. Note that the proof is very similar to the proof of invariance for Khovanov homology, see [BN02, Section 3.5].
The third claim follows from Lemma 3 of the previous section on page 36, since the chosen framing of the cube flow category ensures that the framed Khovanov flow category refines the Khovanov chain complex. ■

Definition. A full subcategory $\mathscr{C}'$ of a flow category $\mathscr{C}$ is called **downward closed** (resp. **upward closed**) if for all $\mathbf{x} \in \operatorname{ob}\mathscr{C}'$, $\mathcal{M}(\mathbf{x},\mathbf{y}) \neq \emptyset$ (resp. $\mathcal{M}(\mathbf{y},\mathbf{x}) \neq \emptyset$) implies $\mathbf{y} \in \operatorname{ob}\mathscr{C}'$.

It is clear that downward/upward closed subcategories are flow categories again.

Lemma 1. Let $\mathscr{C}'$ be a downward closed subcategory of a flow category $\mathscr{C}$ and let $\mathscr{C}''$ be the complementary (upward closed) subcategory of $\mathscr{C}'$ in $\mathscr{C}$. Suppose $C^\bullet(\mathscr{C}'')$ is acyclic. Then the inclusion $|\mathscr{C}'| \hookrightarrow |\mathscr{C}|$ induces a stable homotopy equivalence. (Use the same constants in the construction of $|\mathscr{C}'|$ and $|\mathscr{C}|$.)

Proof. (see [LS11, Lemma 3.32]) Since $C^\bullet(\mathscr{C}'')$ is the quotient of $C^\bullet(\mathscr{C})$ and $C^\bullet(\mathscr{C}')$, we have a short exact sequence

$$0 \to C^\bullet(\mathscr{C}'') \to C^\bullet(\mathscr{C}) \to C^\bullet(\mathscr{C}') \to 0.$$

By Lemma 3 in the previous section (page 36), this is equal to the short exact sequence

$$0 \to \tilde{C}^\bullet(|\mathscr{C}''|) \to \tilde{C}^\bullet(|\mathscr{C}|) \to \tilde{C}^\bullet(|\mathscr{C}'|) \to 0$$

modulo some degree shift. This gives rise to a long exact sequence of reduced cohomology

$$\cdots \to \tilde{H}^{i-1}(|\mathscr{C}'|) \to \tilde{H}^i(|\mathscr{C}''|) \to \tilde{H}^i(|\mathscr{C}|) \to \tilde{H}^i(|\mathscr{C}'|) \to \tilde{H}^{i+1}(|\mathscr{C}''|) \to \cdots$$

where the maps $\tilde{H}^i(|\mathscr{C}|) \to \tilde{H}^i(|\mathscr{C}'|)$ are induced by the inclusion $|\mathscr{C}'| \hookrightarrow |\mathscr{C}|$. Now, $C^\bullet(\mathscr{C}'')$ is acyclic, so $\tilde{H}^i(|\mathscr{C}''|) = 0$ for all i; the same is true for the homology groups of $|\mathscr{C}''|$ by the Universal Coefficient Theorem. So the maps $\tilde{H}_i(|\mathscr{C}'|) \to \tilde{H}_i(|\mathscr{C}|)$ in the long exact sequence of homology are isomorphisms. (This also holds for the unreduced homology groups.) The claim then follows from the following version of Whitehead's Theorem:

> **Theorem** (Whitehead)**.** A map $f : X \to Y$ between simply-connected CW complexes is a homotopy equivalence if $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for each n. (see [H01, p. 367, Corollary 4.33])

In order to find a way around the simply-connectedness hypothesis, we suspend the spaces once and we are done. ■

Lemma 2. Let L be a link diagram obtained by performing a Reidemeister I move on a link diagram L' as in Figure 10 (a). Then $\mathcal{X}^i_{Kh}(L)$ and $\mathcal{X}^i_{Kh}(L')$ are stably homotopy equivalent. (Here, we use the restrictions of the neat embedding and framing of $\mathscr{C}^i_K(L)$ for $\mathscr{C}^i_K(L')$ and the same constants in the constructions of $|\mathscr{C}^i_K(L)|$ and $|\mathscr{C}^i_K(L')|$.)

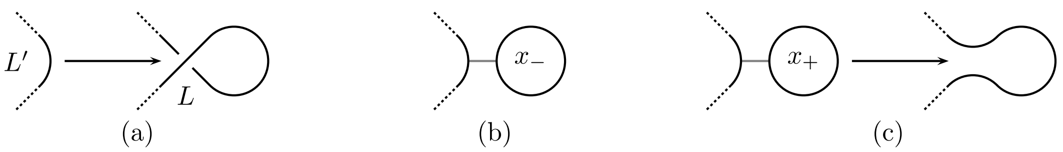
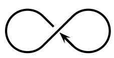


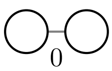
Figure 10: (a) Reidemeister I move and the subcategories (b) $\mathscr{C}'$ and (c) $\mathscr{C}''$

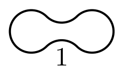
Proof. Let $\mathscr{C} := \mathscr{C}^i_K(L)$. Then $\mathscr{C}^i_K(L')$ is isomorphic to the downward closed subcategory $\mathscr{C}'$ of $\mathscr{C}$ consisting of those diagrams which look like the one in Figure 10 (b). Note that the grading is also preserved, because n_+ increases by one from L' to L (for both possible orientations) and $|x|$ decreases by one because of the x_- in the additional circle. Then the complementary upward closed subcategory $\mathscr{C}''$ consists of exactly those diagrams as in Figure 10 (c). Obviously, $C^\bullet(\mathscr{C}'')$ is acyclic. Hence, by the previous lemma, $|\mathscr{C}|$ and $|\mathscr{C}'|$ are homotopy equivalent. ■

Example. We want to compute the CW complex in a simple case, namely the following oriented one-crossing unknot diagram



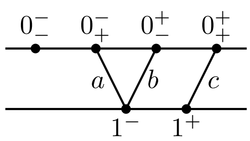
There are six different labellings of the two resolution configurations



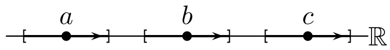
and


,

namely the labellings $\{x_-x_-\}, \{x_+x_-\}, \{x_-x_+\}$ and $\{x_+x_+\}$ of the 0-resolution, denoted by $0^-_-, 0^+_-, 0^-_+$ and 0^+_+ respectively, and $\{x_-\}$ and $\{x_+\}$ of the 1-resolution, denoted by 1^- and 1^+ respectively. The partial order on these labelled resolution configurations is shown in the following picture:



We only need to neatly embed one moduli space, namely the $\langle 0 \rangle$-manifold $\mathcal{M}(1,0)$, which consists of three points. Hence we set $\mathbf{d} := (d_i)$, where $d_i = 1$ for $i = 0$ and 0 otherwise, and choose an embedding $\iota : \mathcal{M}(1,0) \to \mathbb{E}^{\mathbf{d}}[1:0]$, which is just three pairwise distinct points a, b and c on the real line. The framing of these three points must be positive, since $s(\mathcal{C}_{(1),(0)}) = 0$:



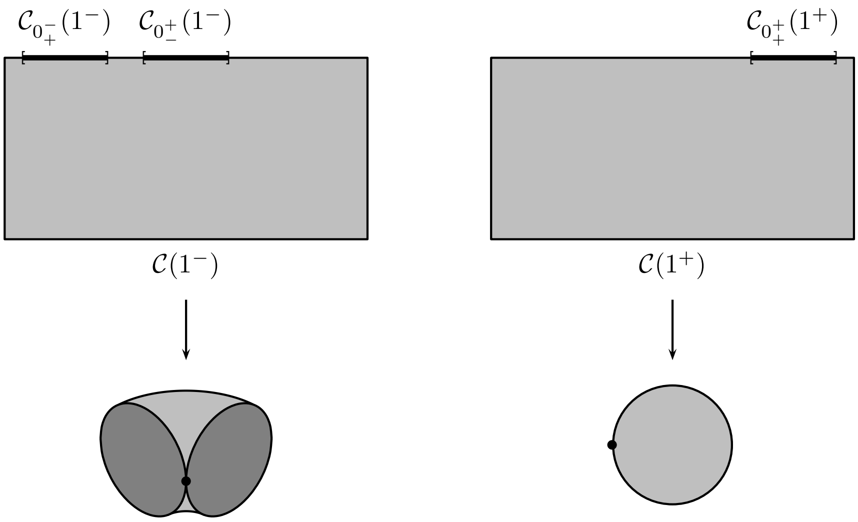
Let $A = 1$ and $B = 0$. Then $C = 1$ and the CW complex has

- one 0-cell (basepoint),
- four 1-cells $\mathcal{C}(0^-_-), \mathcal{C}(0^+_-), \mathcal{C}(0^-_+)$ and $\mathcal{C}(0^+_+)$, which are copies of $[-\varepsilon, \varepsilon]$ and
- two 2-cells $\mathcal{C}(1^-)$ and $\mathcal{C}(1^+)$, which are copies of $[0, R] \times [-R, R]$.

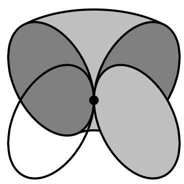
Since the boundary of the four 1-cells is attached to the basepoint, the 1-skeleton of $|\mathscr{C}|$ is just a wedge product of four circles. Next consider the 2-cells. We have

$\mathcal{C}_{0^-_+}(1^-) = \{0\} \times [a-\varepsilon, a+\varepsilon]$, $\mathcal{C}_{0^+_-}(1^-) = \{0\} \times [b-\varepsilon, b+\varepsilon]$ and $\mathcal{C}_{0^+_+}(1^+) = \{0\} \times [c-\varepsilon, c+\varepsilon]$.

Therefore, the 2-cells $\mathcal{C}(1^-)$ and $\mathcal{C}(1^+)$ look like this:



So we get



Hence

$$\mathcal{X}_{Kh}(\infty) = \Sigma^{-1}(S^1_{-1} \vee S^1_{+1} \vee D^1_{+3}) = \Sigma^{-1}(S^1_{-1} \vee S^1_{+1}) = S^0_{-1} \vee S^0_{+1},$$

where the subscripts denote the quantum grading.

3 Steenrod Squares on Khovanov Homology

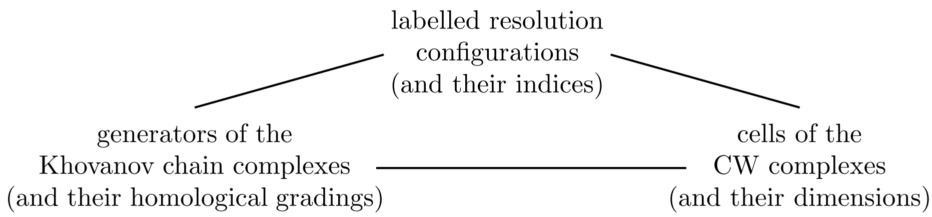
3.1 Motivation

In the previous section, we saw that Khovanov homology could be interpreted as the cohomology of a suspension spectrum. There is a very natural questions to ask here:

> *Does the Khovanov homotopy type constructed above contain more information than Khovanov homology, i. e. do two links exist with the same Khovanov homology, but different Khovanov homotopy types?*

As we can guess already from the example of the one-crossing unknot, the spaces become pretty complicated. Also, in the view of the complexity of this construction, one might ask oneself, whether there is an easier construction. We now want to discuss these questions in more detail from a more general point of view.

Review. Lipshitz and Sarkar's construction is certainly not the only way to obtain spaces for knots and links which return the Khovanov homology invariants as their cohomology groups. For example, adopting a very naïve viewpoint, it would be much easier to just consider the Khovanov homology groups and to construct a space by taking the wedge product of corresponding Moore spaces. (Moore spaces have vanishing reduced homology in all dimensions except for one, where we have one particular group. For example, the spheres are Moore spaces for the group $\mathbb{Z}$.) This space would certainly be an invariant of the link, since Khovanov homology is one. However, this would not be very interesting. So it must have been quite a relief after going through all these efforts, when Lipshitz and Sarkar were able to show that their construction indeed yielded something more sophisticated [LS12, Theorem 1]. Another construction of a Khovanov homotopy type (in a different sense, however) was not successful, see [ELST12].
One central idea in Lipshitz and Sarkar's construction was to start with some kind of "triangle correspondence":



This correspondence simplified the relation between the constructed space and Khovanov homology, since we could use cellular cohomology.
However, for the construction of a CW complex we also needed some recipe to tell us how to glue the cells together, which is why we introduced the Khovanov flow category. In general, there are many ways to assemble a number of given cells such that the resulting CW complexes have certain homology or cohomology groups – and the homotopy types of the various CW complexes do not need to agree.

Example. Consider the standard CW complex structure on the 2-dimensional complex projective space $\mathbb{C}P^2$ consisting of one 2-cell attached to the basepoint by the constant map and one 4-cell with $S^3 \to \mathbb{C}P^1 \cong S^2$ being its attaching map. The integral homology and cohomology groups are therefore $\mathbb{Z}$ in dimensions 0, 2 and 4, and 0 elsewhere.

The wedge product $S^4 \vee S^2$ of the 2- and 4-dimensional spheres has the same homology and cohomology groups. However, $S^4 \vee S^2$ and $\mathbb{C}P^2$ are not homotopy equivalent. This can be shown by using the ring structure on cohomology given by the cup product. Whilst $H^*(\mathbb{C}P^2) \cong \mathbb{Z}[X]/(X^3)$, X being a generator of $H^2(\mathbb{C}P^2)$, the cup product square of a generator of $H^2(S^4 \vee S^2)$ vanishes, so the ring structures on the cohomology groups do not agree and the two spaces cannot be homotopy equivalent.

Refined Tools. As the above example illustrates, the cup product on cohomology can be considered as a refined tool for distinguishing spaces. To get an even better refinement, one constructs so-called cohomology operations, which, in their full generality, are functions

$$\vartheta_X : H^n(X, G) \to H^m(X, H),$$

satisfying the naturality condition $\vartheta_X f^* = f^* \vartheta_Y$ for any continuous map $f : X \to Y$, where G and H are some (abelian) groups. The set of all cohomology operations for the tuple (G, n, H, m) is denoted by $\mathcal{O}(G, n, H, m)$. The cup product gives rise to an example of such an operation, namely by the squaring operation

$$\begin{array}{rcl} H^i(X, G) & \to & H^{2i}(X, G) \\ u & \mapsto & u \cup u \end{array}$$

used in the example above. However, these maps are not compatible with suspensions: If $u \in H^i(X, G)$ and $f_j : H^j(X, G) \to H^{j+1}(\Sigma X, G)$ is the suspension isomorphism, then $f_i(u) \cup f_i(u) \in H^{2i+2}(\Sigma X, G)$, but $f_{2i}(u \cup u) \in H^{2i+1}(\Sigma X, G)$, so the cup square and the suspension isomorphism cannot commute for stupid reasons.
In 1947, Steenrod first described generalisations of this squaring operation, which are now called Steenrod squares. They are functions

$$Sq^i : H^j(X, \mathbb{Z}_2) \to H^{j+i}(X, \mathbb{Z}_2)$$

and their names come from the fact that for $i = j$, Sq^j agrees with the cup product square described above, i. e. $Sq^j(u) = u \cup u$ for $u \in H^j(X, \mathbb{Z}_2)$. (There are also analogous operations with coefficient groups $\mathbb{Z}_p$ for p odd primes, but we do not discuss these here.) It turns out that the Steenrod squares are compatible with suspensions. One also says, the Steenrod squares are **stable operations**:

$$Sq^i(f_j(u)) = f_{j+i}(Sq^i(u)) \quad \text{for any } u \in H^j(X, G).$$

Therefore, we can define them on suspension spectra (and spectra obtained by shifting the indices by some integer).

Conclusion. Using Steenrod operations, we can hope to find an answer to the question whether Lipshitz and Sarkar's construction of a Khovanov homotopy type enables us to distinguish links which Khovanov homology alone cannot. Lipshitz and Sarkar gave a partial answer to this question by finding an example of two links which had isomorphic Khovanov homology groups in one grading, but whose corresponding spectra in this grading were not stably homotopy equivalent [LS12, p. 34, question 5.1]. They also described how the first two Steenrod squares determine the homotopy type for links with reasonably simple Khovanov homology (see the final remark in Section 3.4). Thus, they were able to show that their construction could not be simplified by just looking at Khovanov homology and taking a wedge product of Moore space. In [S12], Seed gave a complete answer to this question based on Lipshitz and Sarkar's formulas for Sq^1 and Sq^2, and the answer is indeed "yes".

Where does this extra information of the Khovanov homotopy type come from? The chain complex $KC^\bullet$ itself probably does not contain significantly more information than its homology groups: Suppose, we have two links whose Khovanov homology groups coincide in all gradings. Then the corresponding chain complexes are chain homotopy equivalent, see for example the discussion at [MO]. So, unless there exists a more subtle relation between chain complexes corresponding to link diagrams of the same link than just chain homotopy equivalence, the extra information has to come solely from the specification of the generators of the chain complexes and the additional structure on these generators, namely the cube and poset structure and the ladybug matchings. In [LS12], Lipshitz and Sarkar describe the first two Steenrod operations in terms of this extra structure, see Sections 3.3 and 3.4.

3.2 Computing Steenrod Squares

In this section, we recall some general facts used in Lipshitz and Sarkar's calculation of the first two Steenrod squares Sq^1 and Sq^2 on Khovanov homology as described in their second paper [LS12]. At the end of this section, we explain the starting point of Lipshitz and Sarkar's calculation of Sq^2. We state their result in Section 3.4. Sq^1 is much easier to compute; we give a detailed description of this computation in Section 3.3.

Properties of the Steenrod Squares. As discussed in the previous section, Steenrod squares are examples of a more general concept, namely cohomology operations. They are maps

$$Sq^i : H^j(X, \mathbb{Z}_2) \to H^{j+i}(X, \mathbb{Z}_2),$$

where $i \in \mathbb{Z}^{\geq 0}$ and $j \in \mathbb{Z}$. For the construction of the Steenrod operations, we refer the reader to [H01, pp. 501f] or [MT68, Section 2]. We just state some of their properties:

1. Sq^i is natural.
2. Sq^i is are stable cohomology operations (in the sense of the previous section).
3. If $u \in H^i(X, \mathbb{Z}_2)$, then $Sq^i(u) = u \cup u = u^2$.
4. If $i > j$, $Sq^i(u) = 0$ for all $u \in H^j(X, \mathbb{Z}_2)$.
5. Sq^0 is the identity.
6. Sq^1 is the Bockstein homomorphism corresponding to the short exact sequence
$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0.$$
7. $Sq^i(u + v) = Sq^i(u) + Sq^i(v)$, i. e. Sq^i is a group homomorphism.
8. $Sq^i(u \cup v) = \sum_{j=0}^{i} Sq^j(u) \cup Sq^{i-j}(v)$ (Cartan formula)
9. For $a < 2b$, $Sq^a Sq^b = \sum_{c=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c \mod 2$ (Adem relations)

Remark. The first property says that Sq^i is a cohomology operation. Property 6 makes the calculation of the first Steenrod squares more or less straightforward, see Section 3.3. The squares are not ring homomorphisms in general. However, if we define

$$Sq(u) := \sum_i Sq^i(u) \quad \text{for } u \in H^i(X, \mathbb{Z}_2),$$

and extend it linearly to non-homogeneous elements $u \in H^*(X, \mathbb{Z}_2)$, the Cartan formula tells us that this is a ring homomorphism. Note that the sum on the right-hand side of the Adem relations is over compositions $Sq^i Sq^j$, where $i \geq 2j$. Indeed, $2a + 2b > 3a \geq 2 \cdot 3c$, so $a + b \geq 3c$, i. e. $a + b - c \geq 2c$, see Theorem 4.

Definition. In analogy with Moore spaces, we define **Eilenberg-MacLane spaces** to be CW complexes which have vanishing homotopy groups in all dimensions except in one. We denote them by $K(G, n)$, where $\pi_n(K(G, n)) = G$. This notation is justified since $K(G, n)$ is uniquely determined by G and n up to homotopy equivalence by a corollary of Theorem 1 below.

Examples. $K(\mathbb{Z}, 1) = S^1$, which follows for example from the theory of covering spaces (see [H01, Section 4.1, pp. 339f]). Using the long exact homotopy sequence of the fibration $S^1 \hookrightarrow S^\infty \twoheadrightarrow \mathbb{C}P^\infty$, one can then show that $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$.

Theorem 1. There is a 1:1-correspondence

$$[X, K(G, n)] \longleftrightarrow H^n(X, G) \quad \text{given by the map} \quad [\vartheta] \mapsto \vartheta^*(\iota_n),$$

where ι_n is a certain class in $H^n(K(G, n), G)$. We want to make this precise, using

Theorem 2 (Hurewicz)**.** Let X be an $(n-1)$-connected space, i. e. $\pi_k(X) = 0$ for all $k < n$. Then the Hurewicz map

$$\varphi : \pi_n(X) \to H_n(X), \quad [f : S^n \to X] \mapsto f_*(1),$$

where 1 is a generator of $H_n(S^n) \cong \mathbb{Z}$, is an isomorphism for $n \geq 2$ and an epimorphism for $n = 1$ with the kernel of φ being the commutator of $\pi_n(X)$.

If we apply this to the $(n-1)$-connected space $K(G, n)$, we get an isomorphism

$$G = \pi_n(K(G, n)) \to H_n(K(G, n))$$

for every n (including $n = 1$, since G is abelian). Denote its inverse by ι. By the Universal Coefficient Theorem, we also have an isomorphism

$$H^n(K(G, n), G) \to \text{Hom}(H_n(K(G, n)), G).$$

Then ι_n is the element on the left-hand side corresponding to ι under this isomorphism. We call ι_n the **fundamental class**.

Corollary. $K(G, n)$ is uniquely determined by G and n up to homotopy equivalence.

Proof. See [MT68, p. 3]. Whitehead's Theorem (page 41) is the central ingredient. ■

Construction of Eilenberg-MacLane Spaces. We want to construct $K(G,n)$, for an abelian group G and $n \in \mathbb{Z}^{>0}$. We are going to apply this construction for $G = \mathbb{Z}_2$ in the calculation of the Steenrod squares below. The main reference is [MT68, p. 5].
Take a free resolution $0 \to E \to F \to G \to 0$ of G. (This is possible, since every subgroup of a free abelian group is free.) Let $\{e_j\}$ be a set of generators of E and likewise $\{f_i\}$ a set of generators of F. Then define the n-skeleton K^n of $K = K(G,n)$ to be a wedge product of n-spheres, one for each generator f_i of F. The $(n+1)$-skeleton K^{n+1} is obtained by attaching one $(n+1)$-cell for each generator e_j of E according to the map $E \to F$.
By Hurewicz's Theorem, $\pi_n(K^{n+1}) \cong H_n(K^{n+1}) = \operatorname{coker}(E \to F) = G$. It remains to get rid of the higher homotopy groups. We do this by transfinite induction, using the following lemma.

Lemma. Let X be a CW complex and $f : S^n \to X$ representing a class in $\pi_n(X)$. Attach an $(n+1)$-cell e to X via f and call the resulting CW complex $Y = X \cup_f e$. Then the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_k(X) \to \pi_k(Y)$ for $k < n$ and an epimorphism $\pi_n(X) \to \pi_n(Y)$, taking the class represented by f to 0.

Proof. This follows from the Cellular Approximation Theorem and the long exact homotopy sequence of the pair (Y, X) (see [H01, p. 351]). ∎

For the construction of the Steenrod squares, we want to better understand the Eilenberg-MacLane spaces for $G = \mathbb{Z}_2$. In fact, the cohomology of Eilenberg-MacLane spaces classifies all cohomology operations:

Theorem 3. [MT68, p. 4] There is a 1:1-correspondence between cohomology operations $\mathcal{O}(G,n,H,m)$ and $H^m(K(G,n),H)$ given by the map which sends a cohomology operation ϑ to $\vartheta_{K(G,n)}(\iota_n)$, where ι_n is the fundamental class in $H^n(K(G,n),G)$.

Example. For $G = H = \mathbb{Z}_2$, Theorem 3 tells us that every cohomology operation

$$\vartheta : H^n(\cdot, \mathbb{Z}_2) \to H^m(\cdot, \mathbb{Z}_2)$$

is determined by the image of the fundamental class $\iota_n \in H^n(K(\mathbb{Z}_2,n),\mathbb{Z}_2)$ under the map

$$\vartheta_{K(\mathbb{Z}_2,n)} : H^n(K(\mathbb{Z}_2,n),\mathbb{Z}_2) \to H^m(K(\mathbb{Z}_2,n),\mathbb{Z}_2).$$

Remark. Relating Theorem 1 on the previous page to Theorem 3 above gives us the following picture:

$$\mathcal{O}(G,n,H,m) \overset{1:1}{\longleftrightarrow} H^m(K(G,n),H) \overset{1:1}{\longleftrightarrow} [K(G,n),K(H,m)].$$

Definitions. For $I = \{i_1, \dots, i_n\}$, a sequence $Sq^I = Sq^{i_1} \dots Sq^{i_n}$ is called **admissible**, if $i_k \geq 2i_{k+1}$ for all k. Using the Adem relations, all sequences can be written as linear combinations of admissible sequences. The **excess** of an admissible sequence is defined by $e(I) = \sum_{k=1}^{n}(i_k - 2i_{k+1})$, where $i_{n+1} = 0$ by convention. It can be regarded as a measure for "how much Sq^I exceeds being admissible" ([H04, p. 52]).

Using spectral sequences, one can show the following (see [H04, p. 53]):

Theorem 4 (Serre)**.** $H^*(K(\mathbb{Z}_2,n),\mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[Sq^I(\iota_n)]$, where ι_n is the generator of $H^n(\mathbb{Z}_2,n),\mathbb{Z}_2)$ and I ranges over all admissible sequences of excess $e(I) < n$.

Examples. For $n = 1$, the only admissible sequence is $Sq^I = Sq^0$, so $H^*(K(\mathbb{Z}_2, 1), \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\iota_1]$ in one variable ι_1. Since we are interested in the second Steenrod squares, we would like to know the $(n+2)^{\text{nd}}$ cohomology groups. The theorem tells us that $H^{n+2}(K(\mathbb{Z}_2, n), \mathbb{Z}_2) = \mathbb{Z}_2$, generated by $Sq^2(\iota_n)$ for $n \geq 3$. For $n = 2$, the generator of $H^4(K(\mathbb{Z}_2, 2), \mathbb{Z}_2) = \mathbb{Z}_2$ is also $\iota_2^2 = \iota_2 \cup \iota_2 = Sq^2(\iota_2)$ (and for $n = 1$, the generator is ι_1^3).

In Section 3.4, we give an explicit formula for the second Steenrod squares. We now want to explain the starting point of Lipshitz and Sarkar's calculations:

Calculation of Sq^2. First, we give an explicit description of $K_n := K(\mathbb{Z}_2, n)$ as a CW-complex. We start by attaching an n-cell e_n to the basepoint by the constant map. Next, we take an $(n+1)$-cell and attach it to the n-skeleton $K_n^n = S^n$ of K_n via a degree 2 map. Thus, $H_n(K_n^{n+1}) = \mathbb{Z}_2$ and all lower homology groups vanish; hence $\pi_n(K_n^{n+1}) = \mathbb{Z}_2$ by Hurewicz's Theorem. We now have to kill all higher homotopy groups, using the above lemma. Lipshitz and Sarkar show that for $n \geq 3$, $\pi_{n+1}(K_n^{n+1}) = \mathbb{Z}/2$ (see [LS11, p. 10]). So we attach an $(n+2)$-cell via a map $S^{n+1} \to K_n^{n+1}$ representing the non-zero element in $\pi_{n+1}(K_n^{n+1})$. Therefore, the $(n+2)$-skeleton of K_n has exactly one $(n+2)$-cell e_{n+2}.
Now, let $c \in H^n(X, \mathbb{Z}_2)$. Then Theorem 1 tells us that there is an (up to homotopy equivalence) unique map

$$\mathfrak{c} : X \to K_n$$

such that $\mathfrak{c}^*(\iota_n) = c$. The fundamental class $\iota_n \in H^n(K_n, \mathbb{Z}_2)$ sends the cycle corresponding to the n-cell e_n to $1 \in \mathbb{Z}_2$, so for any n-cell f_n of X, $c(f_n) = \mathfrak{c}^*(\iota_n)(f_n) = \iota_n(\mathfrak{c} f_n)$ is the degree (modulo 2) of the map

$$S^n \cong f_n/\partial f_n \hookrightarrow X^n/X^{n-1} \to K_n^n = e_n/\partial e_n \cong S^n.$$

By naturality, we have

$$Sq^2(c) = Sq^2(\mathfrak{c}^*(\iota_n)) = \mathfrak{c}^*(Sq^2(\iota_n)).$$

We know the second square $Sq^2(\iota_n)$ (for example by Theorem 4 above): $Sq^2(\iota_n)$ sends the cycle corresponding to the $(n+2)$-cell e_{n+2} to $1 \in \mathbb{Z}_2$. So it remains to understand the map $\mathfrak{c}$.
We can simplify the situation: The inclusion $X^{n+2} \hookrightarrow X$ induces an inclusion

$$H^{n+2}(X, \mathbb{Z}_2) \hookrightarrow H^{n+2}(X^{n+2}, \mathbb{Z}_2)$$

(for example by the long exact sequence of the pair (X, X^{n+2})), so $Sq^2(c)$ is already determined by its restriction to $H^{n+2}(X^{n+2}, \mathbb{Z}_2)$. Hence, it suffices to construct a map $\mathfrak{c} : X^{n+2} \to K_n$. Furthermore, by cellular approximation, we may assume that this map is cellular, so we have in fact a map $\mathfrak{c} : X^{n+2} \to K_n^{n+2}$. Then for every $(n+2)$-cell f_{n+2} of X,

$$\begin{aligned} Sq^2(c)(f_{n+2}) &= \mathfrak{c}^*(Sq^2(\iota_n))(f_{n+2}) = Sq^2(\iota)(\mathfrak{c} f_{n+2}) \\ &= Sq^2(\iota)(d_{\mathfrak{c}(f_{n+2}, e_{n+2})} \cdot e_{n+2}) = d_{\mathfrak{c}(f_{n+2}, e_{n+2})} \cdot Sq^2(e_{n+2}) = d_{\mathfrak{c}(f_{n+2}, e_{n+2})}, \end{aligned}$$

where $d_{\mathfrak{c}(f_{n+2}, e_{n+2})}$ is the degree (modulo 2) of the map

$$S^{n+2} \cong f_{n+2}/\partial f_{n+2} \hookrightarrow X^{n+2}/X^{n+1} \to K_n^{n+2}/K_n^{n+1} \cong S^{n+2}.$$

The major part of the paper [LS12] deals with the construction of this map $\mathfrak{c}$. However, in addition to being quite lengthy, it requires detailed analysis of the framing of the cube flow category, which we do not discuss in this essay.

3.3 Sq^1 on Khovanov Homology

The Bockstein Homomorphism. Sq^1 is the Bockstein homomorphism (that is the connecting map) of the long exact sequence of cohomology induced by the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\cdot 2} \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Let us briefly recall its construction. If $C^\bullet$ is a cochain complex, we obtain a short exact sequence

$$0 \longrightarrow C^\bullet \otimes \mathbb{Z}_2 \xrightarrow{\cdot 2} C^\bullet \otimes \mathbb{Z}_4 \longrightarrow C^\bullet \otimes \mathbb{Z}_2 \longrightarrow 0,$$

which gives rise to a long exact sequence of cohomology in the usual way. In particular, the connecting map

$$H^i(C^\bullet, \mathbb{Z}_2) \to H^{i+1}(C^\bullet, \mathbb{Z}_2)$$

is defined by the following diagram chase:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^\bullet \otimes \mathbb{Z}_2 & \xrightarrow{\cdot 2} & C^\bullet \otimes \mathbb{Z}_4 & \xrightarrow{\mathbf{c} \leftarrow\!\shortmid\, \mathbf{c}} & C^\bullet \otimes \mathbb{Z}_2 & \longrightarrow & 0 \\
 & & \downarrow \delta & & {\scriptstyle \mathbf{c} \mapsto \delta\mathbf{c}} \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & C^\bullet \otimes \mathbb{Z}_2 & \underset{\frac{\delta \mathbf{c}}{2} \leftarrow\!\shortmid\, \delta \mathbf{c}}{\xrightarrow{\cdot 2}} & C^\bullet \otimes \mathbb{Z}_4 & \longrightarrow & C^\bullet \otimes \mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

We now want to find some explicit formula for this map in the case of the Khovanov chain complex $KC^\bullet$. For this, we introduce some notation.

Definition. Let KC^i and KG^i denote the reductions modulo 2 from now on. For an arbitrary cochain $\mathbf{c} \in KC^i$ and a generator $\mathbf{y} \in KG^i$, write $\mathbf{y} \in \mathbf{c}$ if the coefficient of $\mathbf{y}$ in $\mathbf{c}$ is 1, i. e. not 0. Furthermore, given a generator $\mathbf{x} \in KG^{i+1}$, define

$$\mathcal{G}_{\mathbf{c}}(\mathbf{x}) := \{\mathbf{y} \in KG^i | \mathbf{x} \in \delta \mathbf{y} \text{ and } \mathbf{y} \in \mathbf{c}\}.$$

Note that $\mathbf{x} \in \delta \mathbf{y}$ is equivalent to $\mathbf{y} \prec_1 \mathbf{x}$.

Calculation of Sq^1 on Khovanov Homology. Recall from Section 1.3 the definition of the differential of the Khovanov chain complex. We can rewrite it as

$$\delta \mathbf{y} = \sum_{\mathbf{y} \prec_1 \mathbf{x}} (-1)^{s(\mathcal{C}_{\mathscr{F}(\mathbf{x}), \mathscr{F}(\mathbf{y})})} \mathbf{x},$$

where $\mathscr{F} : \mathscr{C}_K \to \mathscr{C}_C(n)$ is the functor from the construction of the Khovanov flow category, sending each labelled resolution configuration to its corresponding vertex of the cube. For an arbitrary cocycle $\mathbf{c} \in KC^i$, let $[\mathbf{c}]$ be the corresponding class in Kh^i. Then

$$\begin{aligned}
Sq^1([\mathbf{c}]) &= \frac{\delta \mathbf{c}}{2} = \frac{1}{2} \sum_{\mathbf{y} \in \mathbf{c}} \delta \mathbf{y} = \frac{1}{2} \sum_{\mathbf{y} \in \mathbf{c}} \sum_{\mathbf{y} \prec_1 \mathbf{x}} (-1)^{s(\mathcal{C}_{\mathscr{F}(\mathbf{x}), \mathscr{F}(\mathbf{y})})} \mathbf{x} \\
&= \sum_{\mathbf{x} \in KG^{i+1}} \frac{\#\{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x}) | s(\mathcal{C}_{\mathscr{F}(\mathbf{x}), \mathscr{F}(\mathbf{y})}) = 0\} - \#\{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x}) | s(\mathcal{C}_{\mathscr{F}(\mathbf{x}), \mathscr{F}(\mathbf{y})}) = 1\}}{2} \mathbf{x}.
\end{aligned}$$

Since $\mathbf{c}$ is a cocycle, $\#\mathcal{G}_{\mathbf{c}}(\mathbf{x})$ is even for every $\mathbf{x} \in KG^{i+1}$, so the fraction is an element in $\mathbb{Z}_2$. We can interpret this fraction as follows: Choose some matching of the elements in $\mathcal{G}_{\mathbf{c}}(\mathbf{x})$, that is a fix point free involution $\mathfrak{b}_{\mathbf{x}} : \mathcal{G}_{\mathbf{c}}(\mathbf{x}) \to \mathcal{G}_{\mathbf{c}}(\mathbf{x})$. Then define a function $\mathfrak{s}_{\mathbf{x}} : \mathcal{G}_{\mathbf{c}}(\mathbf{x}) \to \mathbb{Z}_2$ such that

$$\{\mathfrak{s}_{\mathbf{x}}(\mathbf{y}), \mathfrak{s}_{\mathbf{x}}(\mathfrak{b}_{\mathbf{x}}(\mathbf{y}))\} = \begin{cases} \{0,1\} & \text{if } s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathbf{y})}) = s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathfrak{b}_{\mathbf{x}}(\mathbf{y}))}) \\ \{0\} & \text{if } s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathbf{y})}) \neq s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathfrak{b}_{\mathbf{x}}(\mathbf{y}))}) \end{cases}. \tag{1}$$

Hence,

$$\mathfrak{s}_{\mathbf{x}}(\mathbf{y}) + \mathfrak{s}_{\mathbf{x}}(\mathfrak{b}_{\mathbf{x}}(\mathbf{y})) = \begin{cases} 1 & \text{if } s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathbf{y})}) = s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathfrak{b}_{\mathbf{x}}(\mathbf{y}))}) \\ 0 & \text{if } s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathbf{y})}) \neq s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathfrak{b}_{\mathbf{x}}(\mathbf{y}))}) \end{cases}.$$

We call the collection of pairs $(\mathfrak{b}_{\mathbf{x}}, \mathfrak{s}_{\mathbf{x}})$ a **boundary matching** for $\mathbf{c}$. Then the fraction in the above expression for $Sq^1([\mathbf{c}])$ is equal to

$$\mathfrak{s}(\mathbf{x}) := \sum_{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x})} \mathfrak{s}_{\mathbf{x}}(\mathbf{y}).$$

With this notation, we finally get our result:

Proposition.

$$Sq^1([\mathbf{c}]) = \sum_{\mathbf{x} \in KG^{i+1}} \mathfrak{s}(\mathbf{x})\mathbf{x}.$$

3.4 Sq^2 on Khovanov Homology

The calculation of the second Steenrod square on Khovanov homology is more involved. For the explicit description, we first introduce some notation.

Definition. Consider the standard CW complex structure on the cube $\mathcal{C}(n) := [0,1]^n$ with cells given by

$$\mathcal{C}_{u,v} := \{x \in [0,1]^n | v \leq x \leq u\}$$

where $u, v \in \{0,1\}^n$ such that $v \leq u$. This gives rise to a cellular chain complex $C^\bullet(\mathcal{C}(n))$. Thus, we can think of the standard sign assignment s as a cochain in $C^1(\mathcal{C}(n), \mathbb{Z}_2)$, sending a 1-cell $\mathcal{C}_{u,v}$ to

$$s(\mathcal{C}_{u,v}) = \varepsilon_1 + \dots + \varepsilon_{i-1}$$

where

$$v = (\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \dots, \varepsilon_n) \quad \text{and} \quad u = (\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_n).$$

We now define the **standard frame assignment** $f \in C^2(\mathcal{C}(n), \mathbb{Z}_2)$. A 2-cell $\mathcal{C}_{u,v}$ is sent to

$$f(\mathcal{C}_{u,v}) = (\varepsilon_1 + \dots + \varepsilon_{i-1})(\varepsilon_{i+1} + \dots + \varepsilon_{j-1}),$$

where

$$\begin{aligned} v &= (\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \dots, \varepsilon_{j-1}, 0, \varepsilon_{j+1}, \dots, \varepsilon_n) \quad \text{and} \\ u &= (\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_{j-1}, 1, \varepsilon_{j+1}, \dots, \varepsilon_n). \end{aligned}$$

Definition. Let $\mathbf{x} = (D_L(u), x) \in KG^{i+2}$ and $\mathbf{y} = (D_L(v), y) \in KG^i$ be some labelled resolution configurations with $\mathbf{x} \succ \mathbf{y}$. Let

$$\mathcal{G}_{\mathbf{x},\mathbf{y}} = \{\mathbf{z} \in KG^{i+1} | \mathbf{x} \succ \mathbf{z} \succ \mathbf{y}\}.$$

By Lemma 2 in Section 1.3, this set consists of two or four elements. In the second case, $D_L(u) \smallsetminus D_L(v)$ is the ladybug configuration. Then the elements in $\mathcal{G}_{\mathbf{x},\mathbf{y}}$ naturally correspond to the four labelled resolution configurations between $D_L(u) \smallsetminus D_L(v)$ and $s(D_L(u) \smallsetminus D_L(v))$. In the construction of the moduli spaces $\mathcal{M}(D, x, y)$ for basic index 2 decorated resolution configurations (D, x, y) (page 31), we defined the ladybug matching on these. This induces a matching of the elements in $\mathcal{G}_{\mathbf{x},\mathbf{y}}$. We denote this matching by $\mathfrak{l}_{\mathbf{x},\mathbf{y}}$ and call it the **ladybug matching**. Again, we can think of it as a fix point free involution of $\mathcal{G}_{\mathbf{x},\mathbf{y}}$. In the case $|\mathcal{G}_{\mathbf{x},\mathbf{y}}| = 2$, we define $\mathfrak{l}_{\mathbf{x},\mathbf{y}}$ to be the unique fix point free involution.

Definition. Let $\mathbf{c} \in KC^i$ be a cycle. Choose a boundary matching $(\mathfrak{b}_{\mathbf{z}}, \mathfrak{s}_{\mathbf{z}})$ for $\mathbf{c}$. For $\mathbf{x} \in KG^{i+2}$, define an edge labelled graph $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ in the following way: The set of vertices of $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ is given by

$$\mathcal{G}_{\mathbf{c}}(\mathbf{x}) = \{(\mathbf{z}, \mathbf{y}) \in KG^{i+1} \times KG^i | \mathbf{x} \in \delta\mathbf{z}, \mathbf{z} \in \delta\mathbf{y}, \mathbf{y} \in \mathbf{c}\}.$$

There is an edge between two element $(\mathbf{z}, \mathbf{y})$ and $(\mathbf{z}', \mathbf{y}')$, iff one of the following applies:

(e-1) $\mathbf{y} = \mathbf{y}'$ and the ladybug matching $\mathbf{l}_{\mathbf{x},\mathbf{y}}$ matches $\mathbf{z}$ and $\mathbf{z}'$. Then the edge is unoriented and labelled by the standard frame assignment $f(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathbf{y})})$.

(e-2) $\mathbf{z} = \mathbf{z}'$ and the matching $\mathfrak{b}_{\mathbf{z}}$ matches $\mathbf{y}$ and $\mathbf{y}'$. Then the edge is labelled by 0. Furthermore, if $\mathfrak{s}_{\mathbf{z}}(\mathbf{y}) = \mathfrak{s}_{\mathbf{z}}(\mathbf{y}')$, the edge is unoriented. Otherwise, the edge is oriented from 0 to 1, i. e. if $\mathfrak{s}_{\mathbf{z}}(\mathbf{y}) = 0$ and $\mathfrak{s}_{\mathbf{z}}(\mathbf{y}') = 1$, the edge is oriented from $(\mathbf{z}, \mathbf{y})$ to $(\mathbf{z}, \mathbf{y}')$.

Lemma. Each component of the graph $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ is an even cycle. Furthermore, the number of oriented edges is even in each cycle.

Proof. Each component of $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ is a cycle since at each vertex $(\mathbf{z}, \mathbf{y})$, there is exactly one $\mathbf{z}' \in KG^{i+1}$ such that $\mathfrak{l}_{\mathbf{x},\mathbf{y}}(\mathbf{z}) = \mathbf{z}'$ and exactly one $\mathbf{y}' \in KG^i$ such that $\mathfrak{b}_{\mathbf{z}}(\mathbf{y}) = \mathbf{y}'$. Alternatingly, either the first or the second entry stays the same along the edges of each cycle, so each cycle is even.
For the second claim, label each vertex $(\mathbf{z}, \mathbf{y})$ by

$$s(\mathcal{C}_{\mathscr{F}(\mathbf{x}),\mathscr{F}(\mathbf{z})}) + s(\mathcal{C}_{\mathscr{F}(\mathbf{z}),\mathscr{F}(\mathbf{y})}) \in \mathbb{Z}_2.$$

By equation (3) in the proof of Lemma 1 in Section 1.3, this labelling changes along each edge of type (e-1). Furthermore, it changes along edges of type (e-2) iff the edge is unoriented: An edge between $(\mathbf{z}, \mathbf{y})$ and $(\mathbf{z}, \mathbf{y}')$ is unoriented iff $\mathfrak{s}_{\mathbf{z}}(\mathbf{y}) = \mathfrak{s}_{\mathbf{z}}(\mathbf{y}')$ and by definition of the boundary matching (1) on the previous page, this is equivalent to

$$s(\mathcal{C}_{\mathscr{F}(\mathbf{z}),\mathscr{F}(\mathbf{y})}) \neq s(\mathcal{C}_{\mathscr{F}(\mathbf{z}),\mathscr{F}(\mathbf{y}')}).$$

Since the cycles are even, the number of edges along which the labelling changes is even; so is the number of edges along which the labelling stays the same. Thus, the number of oriented edges in each component is even. ■

Finally, we can state Lipshitz and Sarkar's formula for calculating the second Steenrod square on Khovanov homology:

Theorem. Let $\mathbf{c} \in KC^i$ be a cycle and $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ a graph corresponding to a boundary matching for $\mathbf{c}$. Then the second Steenrod square is given by the following formula:

$$Sq^2([\mathbf{c}]) = \sum_{\mathbf{x} \in KG^{i+2}} (\#|\mathfrak{G}_{\mathbf{c}}(\mathbf{x})| + f(\mathfrak{G}_{\mathbf{c}}(\mathbf{x})) + g(\mathfrak{G}_{\mathbf{c}}(\mathbf{x})))\, \mathbf{x},$$

where

$$\left.\begin{array}{l} \#|\mathfrak{G}_{\mathbf{c}}(\mathbf{x})| := \text{number of components of } \mathfrak{G}_{\mathbf{c}}(\mathbf{x}) \\ f(\mathfrak{G}_{\mathbf{c}}(\mathbf{x})) := \text{sum of the labels of } \mathfrak{G}_{\mathbf{c}}(\mathbf{x}) \\ g(\mathfrak{G}_{\mathbf{c}}(\mathbf{x})) := \text{number of edges with the same orientation (see below)} \end{array}\right\} \quad \text{mod } 2.$$

For the definition of $g(\mathfrak{G}_{\mathbf{c}}(\mathbf{x}))$, we sort the oriented edges in each cycle into two groups, depending on their orientation. We then pick one group from each cycle and take the number of all edges in these groups. The lemma above tells us that $g(\mathfrak{G}_{\mathbf{c}}(\mathbf{x}))$ does not depend on which groups we have picked.

Final Remark. These explicit descriptions of the first two Steenrod squares do not only enable us to find examples of links which show that Lipshitz and Sarkar's Khovanov homotopy type is indeed a stronger invariant, but they are also already sufficient to determine the constructed homotopy type in a large number of cases.
We know that the homotopy type of Moore spaces $M(G, n)$ is completely determined by G and n, provided $n > 1$ [H01, p. 368]. One can find similar results for spaces with reasonably simple prescribed (co)homology groups. Lipshitz and Sarkar make use of the following:
The homotopy type of any simply connected CW complex whose only torsion in cohomology is 2-torsion and whose reduced cohomology groups are supported in three consecutive gradings is completely determined by the cohomology groups and the first two Steenrod squares [LS12, pp. 31f].
Thus, Lipshitz and Sarkar are able to show that if the Khovanov homology of a link satisfies certain conditions, which are satisfied for all links up to 11 crossings, then the homotopy type is a wedge product of various suspensions of the spaces S^0, $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{R}P^5/\mathbb{R}P^2$, $\mathbb{R}P^4/\mathbb{R}P^1$ and $\mathbb{R}P^2 \wedge \mathbb{R}P^2$.

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