FOAM EVALUATION AND LINK HOMOLOGY THEORIES

LECTURE NOTES FOR “PERSPECTIVES ON QUANTUM LINK HOMOLOGY THEORIES”

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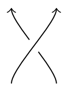
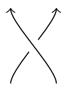
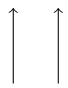
ABSTRACT.

CONTENTS

INTRODUCTION AND DISCLAIMER

1. LECTURE 1: $\mathfrak{gl}_N$-INVARIANTS VIA GRAPH COLORINGS

If H is a quantum group and L is a link colored with finite dimensonial H-modules, Reshetikhin–Turaev associate to this data a Laurent polynomial in q. In this first lecture, we aim to describe these link invariants in term of graph coloring in the case $H = U_q(\mathfrak{gl}_N)$ and the H-module are (quantum) exterior powers of V, the standard representation of H. If $N = 2$, the only relevant representation is V itself and the the invariant is the Jones polynomial. For general N, if all components are colored by V, the invariant is called the $\mathfrak{gl}_N$ *polynomial* and is denoted P_N. It satisfies the following skein relation:

$$q^{-N}P_N\left(\right.$$  $$\left.\right) - q^{N}P_N\left(\right.$$  $$\left.\right) = P_N\left(\right.$$  $$\left.\right).$$

We will actually first focus on a framed version of these invariant: they satisfy a Reidemeister I up to normalization.

The content of this lecture follows [MOY98] and its reinterpretation given in [Rob15].

1.1. **Quantum binomials.** In all lectures, q is a formal parameter. The aim of this section is to give a quantum version of the following identity:

$$\overset{\text{cardinal}}{\#}\overset{\text{powerset}}{\mathscr{P}}_{\underset{\text{only subsets with } a \text{ elements}}{a}}\left(\llbracket \overset{\{1,\dots,n\}}{n}\rrbracket\right) = \underset{a \text{ choose } n}{\binom{n}{a}}.$$

Namely:

$$\overset{\text{quantum cardinal}}{\#_q}\mathscr{P}_a\left(\overset{\text{endow } \llbracket n\rrbracket \text{ with } <}{\llbracket n\rrbracket}\right) = \underset{\text{quantum binomial}}{\begin{bmatrix} n\\ a\end{bmatrix}}.$$

In this formula and everywhere else, q is a formal variable.

For k in $\mathbb{Z}$, define $[k] = \dfrac{q^k - q^{-k}}{q-q^{-1}} = \sum_{i=1}^{k} q^{-k-1+2i} \in \mathbb{Z}[q,q^{-1}]$, if $k \geq 0$, define $[k]! = \prod_{i=1}^{k}[i] \in \mathbb{Z}[q,q^{-1}]$, with the usual convention that an empty product is equal to $1 \in \mathbb{Z}[q,q^{-1}]$. Finally, if $n,a \in \mathbb{Z}$, define

$$\begin{bmatrix} n\\ a\end{bmatrix} = \begin{cases} \displaystyle\prod_{k=1}^{a} \frac{[n+1-k]}{[k]} & \text{if } a \geq 0,\\ 0 & \text{ortherwise.}\end{cases}$$

Remark 1.1.
(1) At this stage it is not clear that $\begin{bmatrix} n\\ a\end{bmatrix}$ belongs to $\mathbb{Z}[q,q^{-1}]$.
(2) For $k \in \{-1,0,1\}$, $[k] = k$;
(3) For $k \in \mathbb{Z}$, $[-k] = -[k]$;
(4) For $n,a \in \mathbb{Z}$, $\begin{bmatrix} n\\ a\end{bmatrix} = (-1)^a \begin{bmatrix} a-n-1\\ a\end{bmatrix}$;
(5) For $n,a \in \mathbb{Z}_{\geq 0}$, $\begin{bmatrix} n\\ a\end{bmatrix} = \dfrac{[n]!}{[a]![n-a]!}$.

Lemma 1.2. *The following identities hold:*

(1) $$[m+n] = q^{-n}[m] + q^{m}[n] = q^{n}[m] + q^{-m}[n]$$

for any m, n in $\mathbb{Z}$.

$$\begin{bmatrix} n \\ a \end{bmatrix} = q^{a} \begin{bmatrix} n-1 \\ a \end{bmatrix} + q^{a-n} \begin{bmatrix} n-1 \\ a-1 \end{bmatrix} = q^{-a} \begin{bmatrix} n-1 \\ a \end{bmatrix} + q^{n-a} \begin{bmatrix} n-1 \\ a-1 \end{bmatrix}. \tag{2}$$

for any n, a in $\mathbb{Z}$.

Proof. Computations left to the reader. □

Corollary 1.3. *Quantum binomials are in $\mathbb{Z}\left[q, q^{-1}\right]$ and respect have a parity properties: exponents appearing are either all even or all odd. Moreover, they are symmetric under $q \mapsto q^{-1}$.*

Sketch of proof. If n is nonnegative, argue by induction on n, using that $\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1$ which follows from the definition for all nonnegative n. If n is negative, use Remark 1.1 (4) and the result for $n \geq 0$. □

Definition 1.4. A *weighted set* is a set X together with a map $w_X \colon X \to \mathbb{Z}$. If X is a finite weighted set, the *quantum cardinal* of X is given by the following formula:

$$\#_q X = \sum_{x \in X} q^{w_X(x)} \in \mathbb{Z}_{\geq 0}\left[q, q^{-1}\right].$$

Exercise 1.5. If X and Y a two finite weighted sets, $X \times Y$ is weighted by declaring that $w_{X \times Y}((x,y)) = w_X(x) + w_Y(y)$. Prove that $\#_q(X \times Y) = \#_q X \#_q Y$.

Definition 1.6. Let $(X, \prec)$ be a finite ordered set and $Y \in \mathscr{P}(X)$. Define the *weight* of Y relatively to X by:

$$\begin{aligned} w(Y) = w_{\mathscr{P}(X)}(Y) :=& \#\{(x,y) \in (X \setminus Y) \times Y \text{ such that } x \prec y\} \\ &- \#\{(x,y) \in (X \setminus Y) \times Y \text{ such that } y \prec x\}. \end{aligned}$$

With the last definition, we see that if X is ordered, then for any nonnegative integer a, $\mathscr{P}_a(X)$ is naturally weighted

Proposition 1.7. *For all a, n in $\mathbb{Z}_{\geq 0}$, one has:*

$$\#_q \mathscr{P}_a\left(\llbracket n \rrbracket\right) = \begin{bmatrix} n \\ a \end{bmatrix}.$$

Idea of the proof. One shows that $\#_q \mathscr{P}_a\left(\llbracket n \rrbracket\right)$ satisfies relation (2) and conclude by induction. □

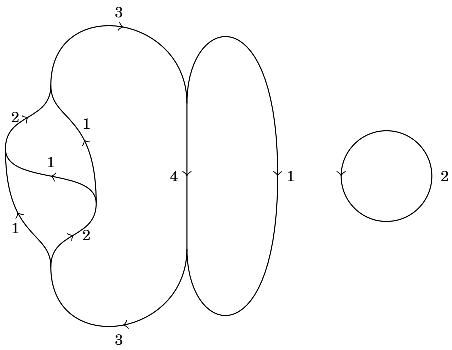
1.2. **MOY graphs.**

Definition 1.8. A *MOY graph*[1] is a trivalent oriented plane[2] graph $\Gamma = (V(\Gamma), E(\Gamma))$ endowed with a thickness function $\ell \colon E(\Gamma) \to \mathbb{Z}_{\geq 0}$ such that the neighborhood of every vertex v is given by one of the two following local models:

The first is called a *split* vertex, the second a *merge* vertex. The edge with thickness $a+b$ (resp. a, resp. b) is called the *thick* (resp. *thin left*, resp. *thin right*) edge at v. This edge is denoted $e_t(v)$ (resp. $e_l(v)$, resp. $e_r(v)$). Vertexless loops are allowed.

[1] MOY stands for **M**urakami–**O**htsuki–**Y**amada. These graphs are often called *webs* in the literature.
[2] Plane means embedded in $\mathbb{R}^2$ rather than embeddable in $\mathbb{R}^2$.

Example 1.9.



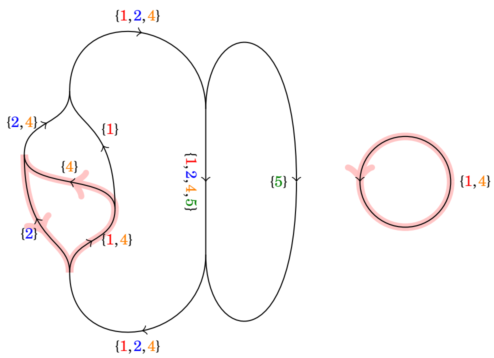
Definition 1.10. Let Γ be a MOY graph. A $\mathfrak{gl}_N$-*coloring* (or simply *coloring*) is a map $c\colon E(\Gamma) \to \mathscr{P}(\llbracket N \rrbracket)$, such that:

(1) For all edge e, $\#c(e) = \ell(e)$.
(2) For every vertex v, $c(e_t(v)) = c(e_l(v)) \sqcup c(e_r(v))$.

The set of $\mathfrak{gl}_N$-colorings of Γ is denoted $\mathrm{col}_{\mathfrak{gl}_N}(\Gamma)$ or simply $\mathrm{col}(\Gamma)$.

For $i < j \in \llbracket N \rrbracket$, denote $\Gamma_{ij}(c)$ union of edges e of Γ such that $\{i,j\} \cap c(e) = 1$. Keep the orientation of edges containing j and reverse the ones containing i. $\Gamma_{ij}(c)$ is an oriented closed curve in $\mathbb{R}^2$.

Example 1.11.



$N = 5$
$\llbracket N \rrbracket = \{1,2,3,4,5\}$
$\Gamma_{2,4}$ is depicted
$w(\Gamma_{2,4}) = 2$

If γ is a finite disjoint union of oriented circles denote $w(\gamma)$ the number of positively oriented circles minus the number of negatively oriented circles in this collection. For c a $\mathfrak{gl}_N$-coloring, define:

$$w(c) = w_{\mathfrak{gl}_N}(c) = \sum_{1 \le i < j \le N} w(\Gamma_{ij}(c)).$$

By this mean, the set $\mathrm{col}_{\mathfrak{gl}_N}(\Gamma)$ is endowed with a weight function.

Definition 1.12. Define the $\mathfrak{gl}_N$ evaluation of a MOY graph Γ by the following formula:

$$\langle \Gamma \rangle_{\mathfrak{gl}_N} = \#_q \mathrm{col}_{\mathfrak{gl}_N}(\Gamma) = \sum_{c \in \mathrm{col}_{\mathfrak{gl}_N}(()\Gamma)} q^{w_{\mathfrak{gl}_N}(c)} \in \mathbb{Z}_{\ge 0}\left[q, q^{-1}\right].$$

Example 1.13.

$$\left\langle \bigcirc\, a \right\rangle_{\mathfrak{gl}_N} = \left\langle \bigcirc\, a \right\rangle_{\mathfrak{gl}_N} = \#_q \mathscr{P}_a\left(\llbracket N \rrbracket\right) = \begin{bmatrix} N \\ a \end{bmatrix}.$$

Lemma 1.14. *(1) $\langle\Gamma\rangle$ is invariant under ambient isotopy.*

(2) For any MOY graph Γ, $\langle\Gamma\rangle$ is symmetric under $q\mapsto q^{-1}$.

(3) For any MOY graph Γ, $\left\langle\overline{\Gamma}\right\rangle=\langle\Gamma\rangle$, where $\overline{\Gamma}$ denotes a mirror image of Γ

(4) For any MOY graphs Γ_1 and Γ_2, $\langle\Gamma_1\sqcup\Gamma_2\rangle=\langle\Gamma_1\rangle\langle\Gamma_2\rangle$.

Sketches of proof. Item (1) is obvious

Item (2) follows from the fact that

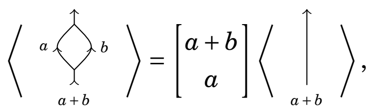
$$\begin{array}{rccc} \iota\colon & [\![N]\!] & \to & [\![N]\!] \\ & k & \mapsto & N+1-k \end{array}$$

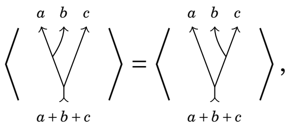
induces an involution on $\mathrm{col}(\Gamma)$ such that $w(\iota(c))=-w(c)$.

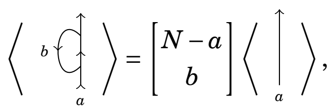
Item (3): there is a natural bijection $\mathrm{col}(\Gamma)\ni c\mapsto\overline{c}\in\mathrm{col}(\overline{\Gamma})$ and $w(\overline{c})=-w(c)$. Conclude with point (2).

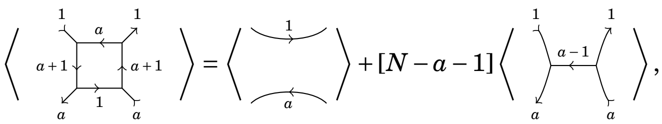
Item (4): there is a natural isomorphism of weighted sets between $\mathrm{col}(\Gamma_1\sqcup\Gamma_2)$ and $\mathrm{col}(\Gamma_1)\times\mathrm{col}(\Gamma_2)$. The result follows from Exercise 1.5. □

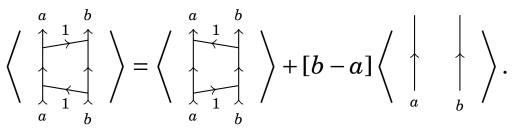
Proposition 1.15. *The following skein theoretic[3] relations hold:*

(3) 

(4) 

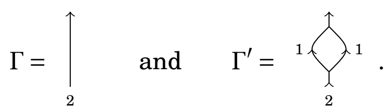
(5) 

(6) 

(7) 

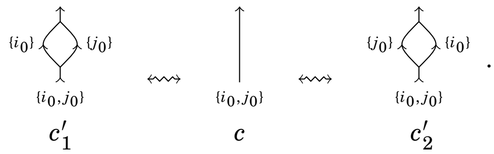
Proof. We only prove some of them in some special cases, the general case and the rest of the relations are left to the reader.

Relation (3) for $a=b=1$. Let us close up both side of the identity and denote Γ the MOY graph on the right-hand side and Γ' that on the left-hand side. Γ and Γ' are equal except in a ball, where:



[3]They hold no matter how one closes everything up.

There is a $2:1$ correspondence between $\mathrm{col}(\Gamma')$ and $\mathrm{col}(\Gamma)$:



To fix the setup, say that $i_0 < j_0$. If, $\{i,j\} \neq \{i_0, j_0\}$, the oriented curves $\Gamma'_{ij}(c'_1)$ and $\Gamma'_{ij}(c'_2)$ are equal and isotopic to $\Gamma_{ij}(c)$. Moreover

$$w\left(\Gamma'_{i_0j_0}(c'_1)\right) = w\left(\Gamma_{i_0j_0}(c)\right) + 1 \qquad \text{and} \qquad w\left(\Gamma'_{i_0j_0}(c'_2)\right) = w\left(\Gamma_{i_0j_0}(c)\right) - 1,$$

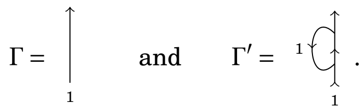
so that

$$w\left(c'_1\right) = w(c) + 1 \qquad \text{and} \qquad w\left(c'_2\right) = w(c) - 1,$$

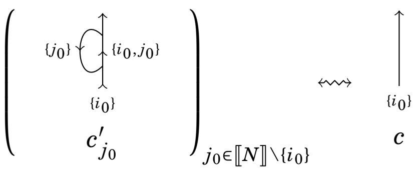
Finally, one has:

$$\langle \Gamma' \rangle = \sum_{c \in \mathrm{col}(\Gamma)} q^{w(c'_1)} + q^{w(c'_2)} = \sum_{c \in \mathrm{col}(\Gamma)} (q + q^{-1}) q^{w(c)} = [2] \langle \Gamma \rangle .$$

Relation (25) for $a = b = 1$. As before we close up both side of the identity and denote Γ the MOY graph on the right-hand side and Γ' that on the left-hand side. Γ and Γ' are equal except in a ball, where:



There is a $N-1:1$ correspondence between $\mathrm{col}(\Gamma')$ and $\mathrm{col}(\Gamma)$:



Let $i < j \in [\![N]\!]$. Unless $\{i,j\} \cap \{i_0, j_0\} \neq \{j_0\}$, one has $w\left(\Gamma_{ij}\left(c'_{j_0}\right)\right) = w(\Gamma_{ij}(c))$ and

$$w\left(\Gamma_{ij_0}\left(c'_{j_0}\right)\right) = \begin{cases} w\left(\Gamma_{ij_0}(c)\right) + 1 & \text{if } i < j_0, \\ w\left(\Gamma_{ij_0}(c)\right) - 1 & \text{if } j_0 > i. \end{cases}$$

Finally, one has:

$$\begin{aligned} \langle \Gamma' \rangle &= \sum_{c \in \mathrm{col}(\Gamma)} \sum_{j_0 \neq i_0} q^{w(c'_{j_0})} = \sum_{c \in \mathrm{col}(\Gamma)} q^{w(c)} \#_q \mathscr{P}_1\left([\![N]\!] \setminus \{i_0\}\right) \\ &= \sum_{c \in \mathrm{col}(\Gamma)} q^{w(c)} \#_q \mathscr{P}_1\left([\![N-1]\!]\right) = [N-1] \langle \Gamma \rangle . \end{aligned}$$

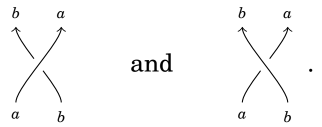
□

Exercise 1.16. Complete the proof of Proposition 1.15.

Theorem 1.17 ([Wu14])**.** *Relations of Proposition 1.15 and Example 1.13 and their mirror images are enough to compute $\langle \Gamma \rangle$ for all Γ. In other words, the $\mathbb{Z}[q, q^{-1}]$-module generated by MOY graphs and modded out by these local relations is 1-dimensional and a base is given by $\emptyset$, and $\Gamma = \langle \Gamma \rangle \emptyset$ in this skein module.*

1.3. **Link invariants.**

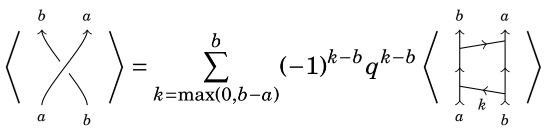
Definition 1.18. A *knotted MOY diagram* D is an oriented plane graph with valency 3 and 4, endowed with a thickness function $\ell\colon E(D) \to \mathbb{Z}_{\geq 0}$ and a sign function $\ell\colon V_4(D) \to \{\pm 1\}$ where $V_4(D)$ is the set of vertices of D of degree 4. The signs function is depicted diagrammatically as for knot diagram. The neighborhoods of all vertex of valence 3 are given by that of MOY graphs. The neighborthood of every vertex of valence 4 is given by one of two following local models:

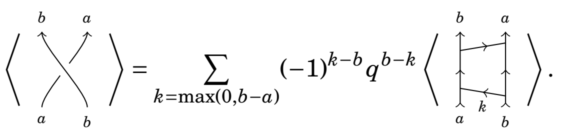


Vertices of degree 4 are called *crossings*.

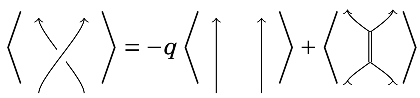
Example 1.19. Oriented link diagrams colored by natural integers and MOY graphs are example of knotted MOY diagrams.

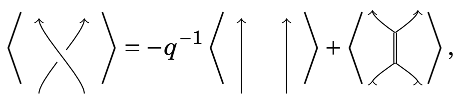
We extend $\langle\cdot\rangle = \langle\cdot\rangle_{\mathfrak{gl}_N}$ to knotted MOY diagram by imposing:

(8)  and

(9) 

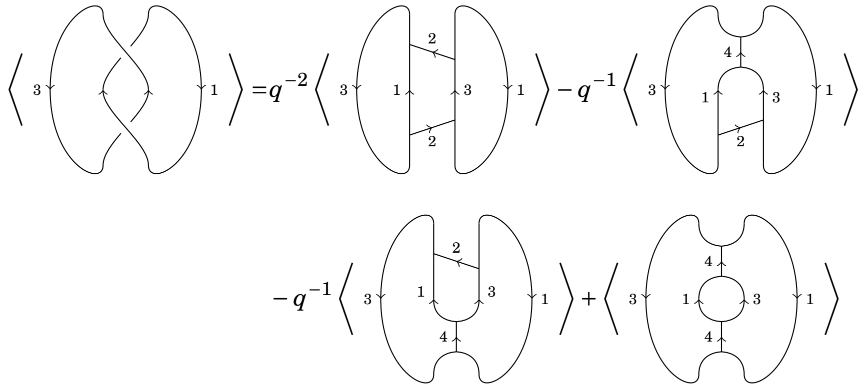
In particular one has:

(10)  and

(11) 

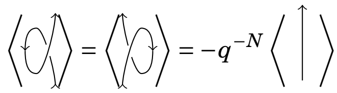
where unlabeled edges are meant to have thickness 1 and edges represented by a double line are meant to have thickness 2.

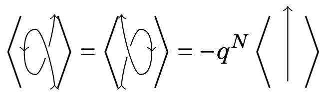
Example 1.20.

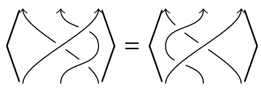


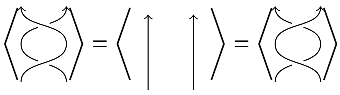
$$=q^{-2}[3][N-2]\begin{bmatrix}N\\3\end{bmatrix}-2q^{-1}[3][4]\begin{bmatrix}N\\4\end{bmatrix}+[4][4]\begin{bmatrix}N\\4\end{bmatrix}.$$

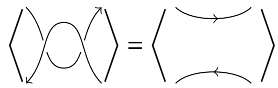
Proposition 1.21. *The following local relations hold:*

(12) 

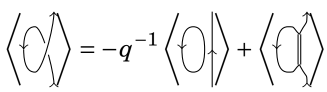
(13) 

(14) 

(15) 

(16) 

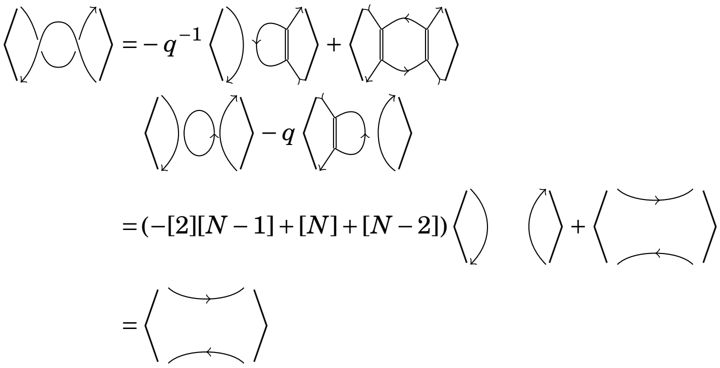
Proof. We only prove some of these identities, the remaining ones are left to the reader.



$$=-q^{-1}[N]\left\langle \uparrow \right\rangle+[N-1]\left\langle \uparrow \right\rangle$$

$$=-q^{-N}\left(q^{N-1}[N]+q^{N}[-N+1]\right)\left\langle \uparrow \right\rangle$$

$$=-q^{-N}\left\langle \uparrow \right\rangle$$



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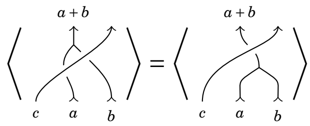
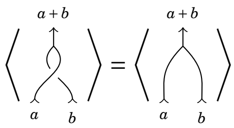
The proofs of the following proposition and its corollary are left to the reader.

Proposition 1.22. *The following local relations hold;*

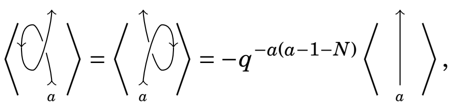
(17) 

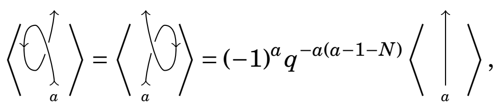
(and 7 analogues),

(18) 

(and 3 analogues).

Corollary 1.23. *The polynomial $\langle\cdot\rangle$ is invariant under colored Reidemeister 2 and Reidemeister 3 moves and satisfies the following local relations:*

(19) 

(20) 

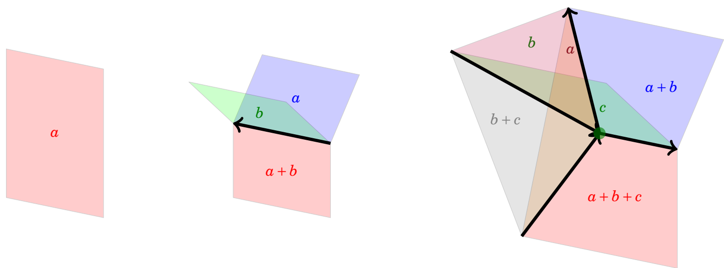
It is therefore an invariant of colored oriented framed links.

Exercise 1.24. Normalise $\langle\cdot\rangle$ to get an invariant of oriented unframed colored links.

2. Lecture 2: Coloring foams

2.1. **Foams and colorings.**

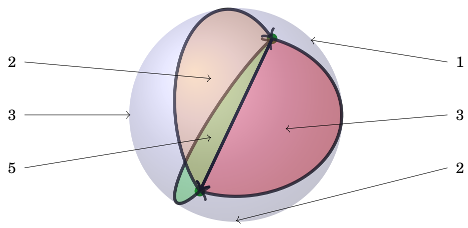
Definition 2.1. A *foam F* is a finite 2-dimensional CW-complex embedded in $\mathbb{R}^3$ endowed with a thickness function $\ell\colon \{\text{2-cells}j\} \to \mathbb{Z}_{\geq 0}$ and orientation of 1- and 2-cells such that the neighborhood of every point has one of the following 3 local models (or their mirror images):



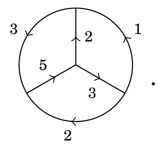
Points with the first (resp. second, resp. third) model are *regular* (resp. *on a binding*, resp. *singular vertex*). Singular vertices (•) and bindings ($\longleftarrow$) form an oriented 4-valent graph denoted $b(F)$ (with possibly vertex-less loops).

The connected components of $F \setminus b(F)$ are called facets should inherit an orientation from the orientation of 2-cells. As for MOY graphs, there are notion of *thin* and *thick* facets at a binding. We impose that orientation of thin facets are consistent with that of the binding and that the orientation of the thick facet to be inconsistent with that of the binding.

Example 2.2. The foam

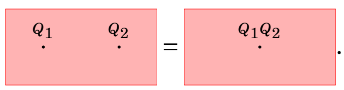


is the double suspension of



Product of $\mathbb{S}^1$ with a MOY graph Γ give an example of a foams without any singular vertices.

Definition 2.3. A *decoration* of a foam F is a map $P\colon \{\text{facets}\} f \mapsto P_f$, where P_f is a symmetric polynomial in $\ell(f)$ variables. We depict decoration pictorially by adding dots labelled by symmetric polynomials on facets. Dots behave multiplicatively:



No dots means the constant polynomial equal to 1. A foam is *decorated* if it is endowed with a decoration.

From now on, all foams are decorated.

Definition 2.4. A $\mathfrak{gl}_N$*-coloring* (or *coloring*) is a foam F is a map $c\colon \{\text{facets}\} \to \mathscr{P}(\llbracket N \rrbracket)$ such that:

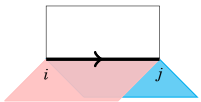
- For every facet f, $\#c(f) = \ell(f)$.

- At every binding, if one denotes f_1 and f_2 the two thin facets and f_t the thick one, one has: $c(f_t) = c(f_1) \sqcup (f_2)$.

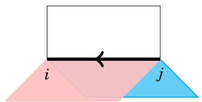
The set of $\mathfrak{gl}_N$-colorings of F is denoted $\mathrm{col}_N(F)$ (or simply $\mathrm{col}(F)$).

Observation 2.5. *Let us consider a foam F and a coloring c.*

1. *Let $i \in [\![N]\!]$. The (closure of the) reunion of facets f of F such that $i \in c(f)$ is an oriented surface. It is denoted $F_i(c)$.*
2. *Let $i < j \in [\![N]\!]$. The (closure of the) reunion of facets f of F such that $\#(c(f) \cap \{i,j\}) = 1$ is an oriented[4] surface. It is denoted $F_{ij}(c)$.*
3. *The surface F_{ij} is partitionned into i- and j-colored regions. These are separated by disjoint circles. Indeed if these circles were to interesct this would happen at singular vertices, but a quick inpsection of what can happen there easily clear this out.*
4. *Each of these circles can be given a sign[5] according to the following convention:*



positive circle



negative circle

The number of positive (resp. negative) circles is denoted $\theta^+_{ij}(F,c)$ (resp. $\theta^-_{ij}(F,c)$). Finally we set $\theta_{ij}(F,c) = \theta^+_{ij}(F,c) + \theta^-_{ij}(F,c)$.

Lemma 2.6. *Given a foam F the quantity*

$$\chi_N(F) = \sum_{1 \le i < j \le N} \chi(F_{ij}(c))$$

does not depend on the $\mathfrak{gl}_N$-coloring c.

Exercise 2.7. Prove Lemma 2.6 by providing a formula independent of coloring.

The $\mathfrak{gl}_N$-*degree* of a decorated foam $(F, P_\bullet)$ is denoted $d_N(F)$ and is given by the following formula:

$$d_N(F) = -\chi_N(F) + 2\sum \deg(()P_f).$$

2.2. **Evaluation.** For what follows we fix $X = \{x_1, \dots, x_N\}$ a set of formal variables. We will be working with polynomial and rational function X and these variables are meant to he homogeneous of degree 2. If $I = \{i_1, \dots i_a\} \subseteq [\![N]\!]$, $X_I = \{x_{i_1}, \dots, x_{i_a}\}$.

Given F a decorated foam and c a coloring, define:

$$\tau(F,c) = \tau_N(F,c) = \frac{(-1)^{s(F,c)} \displaystyle\prod_{f \text{ facet}} P_f(X_{c(f)})}{\displaystyle\prod_{i<j} (x_i - x_j)^{\chi(F_{ij}(c))/2}},$$

where

$$s(F,c) = \sum_{i=1}^{N} i \frac{\chi(F_i(c))}{2} + \sum_{1 \le i < j \le N} \theta^+_{ij}(F,c).$$

[4]Take orientation of facets containing j and reverse the orientation of the one containing i. The orientability of this surface can also be deduced from its embeddability in $\mathbb{R}^3$.

[5]It is possible to remove the embedding in $\mathbb{R}^3$ in the definition of foam, in this case, one should had a cycling ordering of facets around bindings in order to get this signs. A compatibility of these cyclic ordering needs to be imposed at singular vertices.

Finally the $\mathfrak{gl}_N$-*evaluation* of F is given by the following formula:

$$\tau(F) = \tau_N(F) = \sum_{c \in \mathrm{col}()[N]F} \tau_N(F,c).$$

Lemma 2.8. *Let (F,c) be a colored foam, and c' be the coloring of F obtained by swaping i and $i+1$ on c, then:*

$$\tau(F,c) = \tau(F,c')_{|x_i \leftrightarrow x_{i+1}}.$$

Proof. From the formula of $\tau(F,c)$ itself, we get that:

$$\tau(F,c') = (-1)^{s(F,c')-s(F,c)+\chi(F_{i(i+1)}(c))/2}\tau(F,c)_{|x_i \leftrightarrow x_{i+1}}.$$

Hence it is enough to check that $s(F,c') - s(F,c) \equiv \chi(F_{i(i+1)}(c))/2$, where $\equiv$ means equality mod 2. We compute:

$$\begin{aligned}
s(F,c') - s(F,c) &= \frac{i\chi(F_i(c')) + (i+1)\chi(F_{i+1}(c')) - i\chi(F_i(c)) - (i+1)\chi(F_{i+1}(c))}{2} \\
&\quad + \theta^+_{i(i+1)}(F,c') - \theta^+_{i(i+1)}(F,c) \\
&= \frac{i\chi(F_{i+1}(c)) + (i+1)\chi(F_i(c)) - i\chi(F_i(c)) - (i+1)\chi(F_{i+1}(c))}{2} \\
&\quad + \theta^-_{i(i+1)}(F,c) - \theta^+_{i(i+1)}(F,c) \\
&\equiv \frac{\chi(F_i(c)) + \chi(F_{i+1}(c))}{2} + \theta_{i(i+1)}(F,c) \\
&\equiv \frac{2\chi(F_{i\cap(i+1)}(c)) + \chi(F_{i(i+1)}(c))}{2} + \theta_{i(i+1)}(F,c) \\
&\equiv \frac{\chi(F_{i(i+1)}(c))}{2}.
\end{aligned}$$

In this computation, $F_{i\cap(i+1)}$ denotes the oriented surface (with boundary) which is the reunion of facets f such that $\{i,j\} \subseteq c(f)$. The number of boundary component of this surface is precisely $\theta_{i(i+1)}(F,c)$, hence $\chi(F_{i\cap(i+1)}(c)) + \theta_{i(i+1)}(F,c)$ is even. □

Corollary 2.9. *The rational fraction $\tau_N(F)$ is symmetric (in the x_is).*

Example 2.10. Consider the F sphere of thickness 1 decorated by the polynomial y^2 for $N = 3$. F admits three colorings and one has:

$$\begin{aligned}
\tau(F) &= \frac{-x_1^2}{(x_1 - x_2)(x_1 - x_3)} + \frac{-x_2^2}{(x_2 - x_1)(x_2 - x_3)} + \frac{-x_3^2}{(x_3 - x_2)(x_3 - x_1)} \\
&= -1.
\end{aligned}$$

If (F,c) is a colored foam, one can swap i and j along a collection Σ of connected component of $F_{ij}(c)$. This operation is called an (i,j)-*Kempe move along* Σ.

Proposition 2.11 ([?]). *The rational fraction $\tau_N(F)$ is a symmetric polynomial of degree $d_N(F)$.*

The statement about the degree follows directly from the definition of $d_N(F)$ and of $\tau_N(F,c)$ (remember that the x_is have degree 2). First note that

$$\tau(F) \in \mathbb{Z}\left[X, \left(\frac{1}{x_j - x_i}\right)_{1 \le i < j \le N}\right].$$

By symmetry it is enough to show that

$$\tau(F) \in \mathbb{Z}\left[X, \left(\frac{1}{x_j - x_i}\right)_{\substack{1\le i<j\le N\\(i,j)\neq(1,2)}}\right].$$

Before proving this, we'll need two lemmas.

Lemma 2.12. *Suppose that c and c' are related by a $(1,2)$-Kempe move along Σ, then $s(F,c') \equiv s(F,c) + \chi(\Sigma)/2$.*

The proof is a local version of the proof of Lemma2.8.

Lemma 2.13. *Let c anc c' ce two coloring related by a $(1,2)$-Kempe move along Σ and let $3 \le k \in [\![N]\!]$.*

$$\theta^+_{1k}(F,c) + \theta^+_{2k}(F,c) \equiv \theta^+_{1k}(F,c') + \theta^+_{2k}(F,c').$$

Moreover, there exists $t_\Sigma(c,k) \in 2\mathbb{Z}$ such that

$$\chi(F_{1k}(c')) = \chi(F_{1k}) + t_\Sigma(c,k), \quad \textit{and} \quad \chi(F_{2k}(c')) = \chi(F_{2k}) - t_\Sigma(c,k).$$

The integer $t_\Sigma(c,k)$ depends only on how Σ is colored by $1,2$ and k.

The proof of this lemma is left to the reader.

Proof of Proposition 2.11. Consider the equivalence relation on $\mathrm{col}_N(F)$ generated by $(1,2)$-Kempe moves. Let c be a coloring of F. Suppose that $\Sigma = F_{12}(c)$ has r connected components $\Sigma_1,\dots\Sigma_r$, then $[c]$ has 2^r elements. We will that that $\sum_{c'\in[c]}\tau(F,c')$ is a quotient $\frac{A(X)}{B(X)}$ of polynomials in X with $B(X)$ a product of $(x_i - x_j)$ with $(x_1 - x_2)$ not a divisor of $B(X)$, from which the result follows.

Let us introduce some notations:

- Denote $X' := X_{|x_1\leftrightarrow x_2}$.
- For $s \in \{1,\dots,r\}$, denote

$$P_{\Sigma_s}(X) := \prod_{f\subseteq\Sigma_s} P_f(X_{c(f)}) \quad \text{and} \quad \widetilde{P}(X) := \prod_{f\not\subseteq\Sigma} P_f(X_{c(f)})$$

.

- Denote:

$$T_s(X) = P_{\Sigma_s}(X)\prod_{k=3}^{N}(x_1 - x_k)^{t_{\Sigma_s}(c,k)/} + (-1)^{\chi(\Sigma_s)/2}P_{\Sigma_s(X')}\prod_{k=3}^{N}(x_2 - x_k)^{t_{\Sigma_s}(c,k)/2}.$$

- Finally set

$$\widetilde{Q}(X) := \frac{Q(F,c)\prod_{s=1}^{r}\prod_{k\ge3}(x_1 - x_k)^{t_\Sigma(c,k)/2}}{(x_1 - x_2)^{\chi(F_{12}(c))/2}}.$$

  Note that $(x_1 - x_2)$ does not divide $\widetilde{Q}(X)$.

One has:

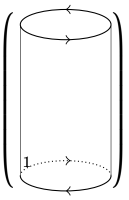
$$\sum_{c\in[c']}\tau(F,c') = (-1)^{s(F,c)}\frac{\widetilde{P}(X)}{\widetilde{Q}(X)}\prod_{s=1}^{r}(x_1 - x_2)^{-\chi(\Sigma_s)/2}T_s(X)$$

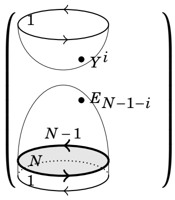
Developing the right-hand side of this identity gives 2^r term each of these corresponding to the contribution of $\tau(F,c')$.

The only problematic factors on the right-hand side are these for whoch Σ_s is a sphere. However, in this case, $T_s(X)$ is antisymetric in x_1 and x_2 and therefore disible by $(x_1 - x_2)$. $\square$

2.3. **Local relations.**

Lemma 2.14 (Neck-cutting)**.** *The polynomial τ satisfy the following relation:*

τ 

$= (-1)^{N(N+1)/2} \sum_{i=0}^{N-1} (-1)^{N-i} \tau$ 

Moreover, for the operation which consist by stacking onto each other, the foams (with boundary) on the right-hand side behave as orthogonal idempotents (up to the sign given in the formula).

Proof. We only prove the local formula. Let us close up the foams identically but arbitrarily and denote by F the foam on the left-hand side and by $(G_i)_{0 \le i \le N-1}$ the ones on the right-hand side and finally by G the foam on the right-hand side with decoration Y^i and E_{N-i-1} removed. The colorings of G (and therefore of all the G_is) are of two types: either they give the same color to top or bottom facets or they do not.

Suppose first that a coloring c give the same color to top and bottom facets. We may suppose that this color is $\{1\}$. In this case this induces a coloring of F (still denoted by c) and we have

$$\tau(G_i, c) = (-1)^{N(N+1)/2} \frac{x_1^i E_{N-1-i}(x_2, \dots, x_N)}{\prod_{k=2}^{N}(x_1 - x_k)} \tau(F, c).$$

On the other hand, one has:

$$\sum_{i=1}^{N-1} (-1)^{N-i} x_1^i E_{N-1-i}(x_2, \dots, x_N) = \prod_{k=2}^{N} (x_1 - x_k),$$

so that

$$(-1)^{N(N+1)/2} \sum_{i=0}^{N-1} (-1)^{N-i} \tau(G_i, c) = \tau(F, c).$$

Suppose now that a coloring c of G give different color on top and bottom. We may suppose that these colors are $\{1\}$ on top and $\{2\}$ on bottom. For such coloring, one has:

$$\tau(G_i, c) = (-1)^{N(N+1)/2+1} x_1^i E_{N-1-i}(x_1, x_3, \dots, x_N) \tau(G, c).$$

However, one has

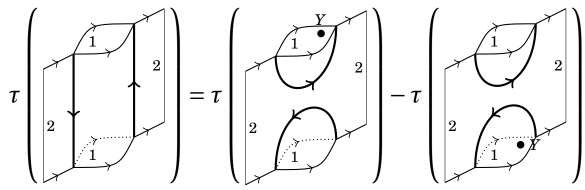
$$\sum_{i=1}^{N-1} (-1)^{N-i} x_1^i E_{N-1-i}(x_1, x_3, \dots, x_N) = \prod_{\substack{k=1 \\ k \neq 2}}^{N} (x_1 - x_k).$$

so that

$$\sum_{i=0}^{N-1} (-1)^{N-i} \tau(G_i, c).$$

$\square$

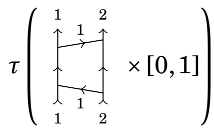
Lemma 2.15 (Digon-cutting)**.** *The polynomial τ satisfy the following relation:*



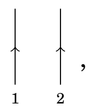
Moreover, for the operation which consist by stacking onto each other, the foam (with boundary) on the right-hand sides behave as orthogonal idempotents (up to the sign given in the formula).

The proof is easier than that for the previous lemma and is left as an exercise.

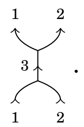
There is an analogue lemma which expresses



as a sum of values of τ on two foams, one factorizes by



the other one by



As before, the two foams behave like orthogonal idempotents.

3. Lecture 3: Universal construction and $\mathfrak{gl}_N$-link homology

3.1. **Universal construction.**

Definition 3.1. Let Γ_0 and Γ_1 be two MOY graphs. A (Γ_0,Γ_1)-*foam* F (denoted $F\colon \Gamma_1 \to \Gamma_0$) is a CW-complex embedded in $\mathbb{R}^2 \times [0,1]$ which looks locally like a foam (with decoration) and such that

$$F \cap \mathbb{R}^2 \times [0,\epsilon) = -\Gamma_0 \times [0,\epsilon)$$
$$F \cap \mathbb{R}^2 \times (1-\epsilon,1] = -\Gamma_1 \times (1-\epsilon,1]$$

The $\mathfrak{gl}_N$-degree of these foams is define by the same formula as that for closed foams.

Foams and MOY graphs fit into a category called Foam, objects are MOY graphs and

$$\mathrm{Hom}(\Gamma_0,\Gamma_1) = \{\Gamma_0,\Gamma_1)\text{-foams}\}/\text{ambient isotopy}.$$

The composition is given by concatenation and re-scaling.

Exercise 3.2. Show that the degree of foams is additive with respect to composition.

We now use an idea, which to the best of our knowledge was first formalized in [BHMV95] and called the *universal construction*. The aim is to promote τ_N into a functor $\mathscr{F}_N\colon \mathsf{Foam} \to R_N\text{-}\mathsf{mod}_{\mathrm{gr}}$, where R_N is the graded ring $\mathbb{Z}[X] = \mathbb{Z}[x_1, \dots, x_N]$ with every x_i homogeneous of degree 2. Grading shifts of graded modules, are denoted by powers of q: q^6M denote the module M where the degree of each element has been increased by 6.

Fix a MOY graph Γ and set:

$$\mathscr{W}_N(\Gamma) = \bigoplus_{F\colon\ \to \Gamma} q^{d_N(F)} R_N.$$

In other words, $\mathscr{W}_N(\Gamma)$ is the infinitely generated graded free R_N-module with an homogeneous base given by $(,\Gamma)$-foams. We now define a bilinear form $(\cdot;\cdot)_N = (\cdot;\cdot)$ on $\mathscr{W}_N(\Gamma)$ on this base:

$$(F;G)_N := \tau_N(\overline{G} \circ F)_N \in R_N$$

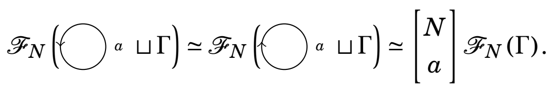
Finally define $\mathscr{F}_N(\Gamma) := \mathscr{W}_N(\Gamma)/\ker(\cdot;\cdot)_N$. This means that $\sum_i \lambda_i F_i = 0 \in \mathscr{F}_N(\Gamma)$ with $F_i\colon \emptyset \to \Gamma$ and $\lambda_i \in R_N$ if and only if $\sum_i \lambda_i \tau_N(G \circ F_i) = 0$ for all $G\colon \Gamma \to \emptyset$.

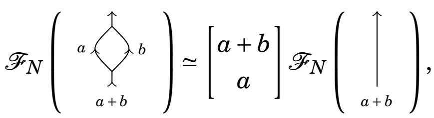
We have defined, $\mathscr{F}_N$ on objects of Foam. The definition extends for free on morphisms. Note that if $F\colon \Gamma_0 \to \Gamma_1$ is a foam, it defines naturally a morphism from $\mathscr{W}_N(\Gamma_0) \to \mathscr{W}_N(\Gamma_1)$.

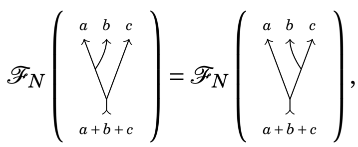
Exercise 3.3. Prove that these morphisms pass to the quotient and define a functor $\mathscr{F}_N\colon \mathsf{Foam} \to R_N\text{-}\mathsf{mod}_{\mathrm{gr}}$.

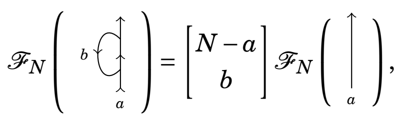
Proposition 3.4. *The functor $\mathscr{F}_N$ satisfies the following local relations and their mirror images. Moreover, the isomorphisms are given by local foams:*

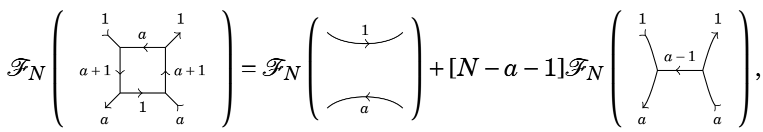
$$\mathscr{F}_N(\emptyset) \simeq R_N. \tag{21}$$

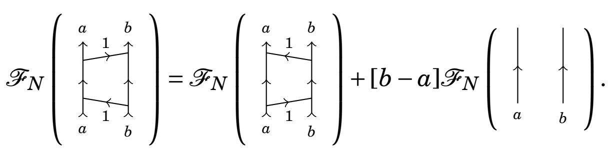
(22) 

(23) 

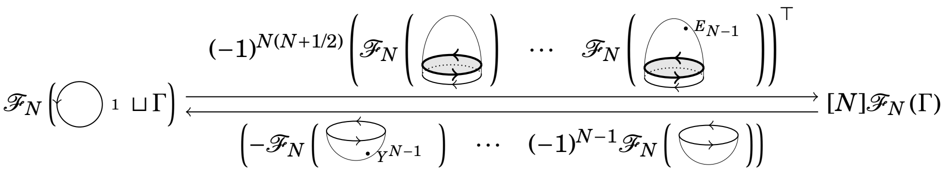
(24) 

(25) 

(26) 

(27) 

Sketch of proof. One uses the local relations of τ. For instance, the pair of morphisms

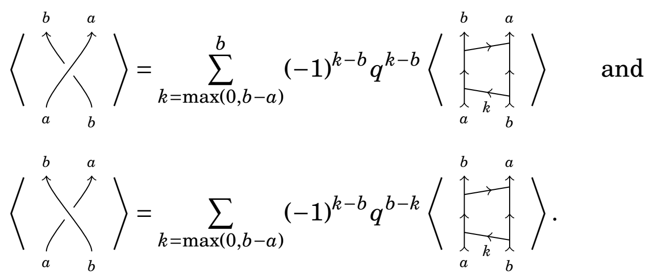


is a pair of mutually inverse isomorphisms. □

Corollary 3.5. *For any MOY graph Γ, $\mathscr{F}_N(\Gamma)$ is projective (and therefore free, because R_N is a polynomial algebra), and its graded rank is $\langle\Gamma\rangle_N$. In other words, $\mathscr{F}_N$ categorifies, $\langle\cdot\rangle_N$.*

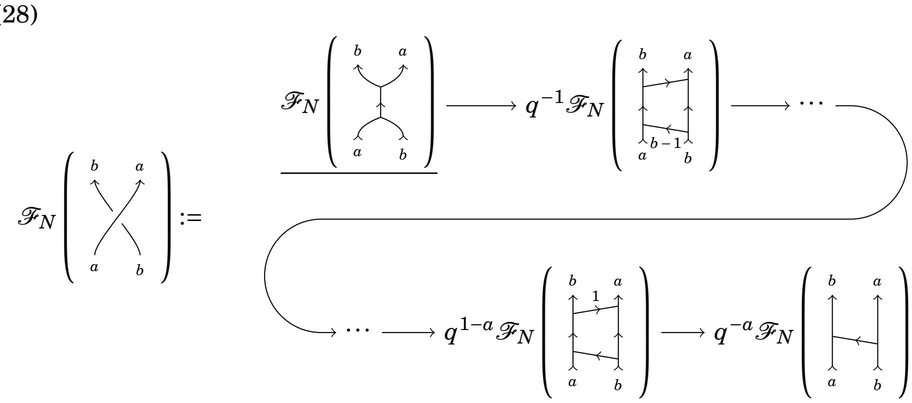
Proof. Theorem 1.17 tells us that any MOY graph can be reduced in finitely many steps to $\emptyset$ using these local relations. We can argue by induction on the number of necessary steps to reduce Γ since a direct factor of a projective module is projective module. This proves that $\mathscr{F}_N(\Gamma)$ is projective. The statement of rank is then obvious: it is a Laurent polynomial in q which satisfies the same local relation as $\langle\cdot\rangle_N$. Theorem 1.17 implies uniqueness of such a quantity. □

3.2. **Rickard complexes and hyper-rectangles.** The aim of this section is to categorify the extension of $\langle\cdot\rangle_N$ given by the relations we imposed:

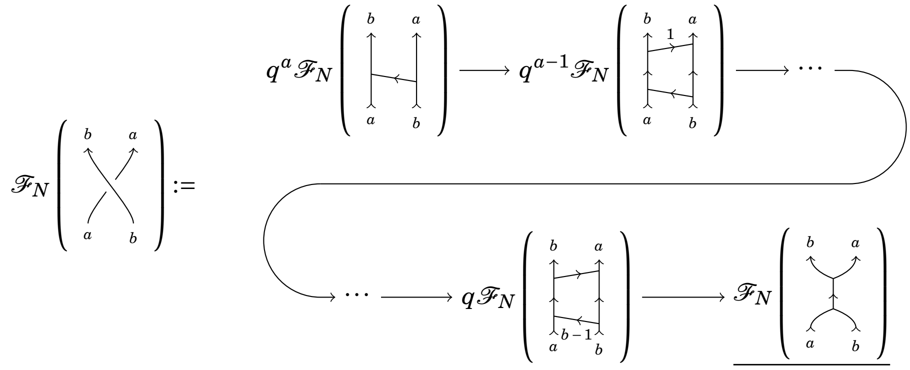


For this we'll define complexes of graded R_N-modules. We do this locally by defining (if $b \geq a$):

(28)



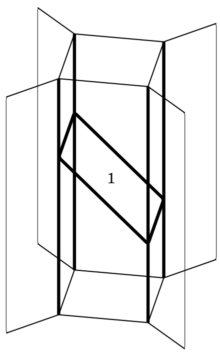
and
(29)

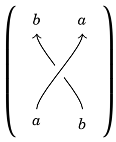


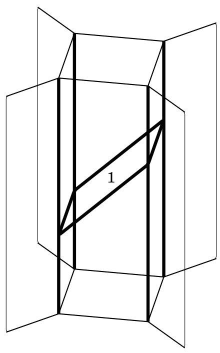
The underline term is meant to be in homological degree 0.

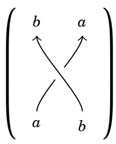
If $a > b$, the complexes are the same, only the last (or first) diagram need to be changed: the rung points right instead of left.

The differentials in this chain complexes are given the images by $\mathscr{F}_N$ of the foams



for $\mathscr{F}_N$  and

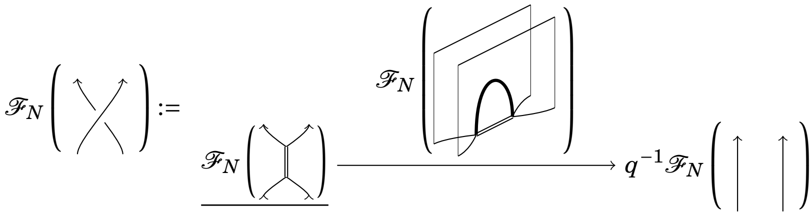


for $\mathscr{F}_N$ .

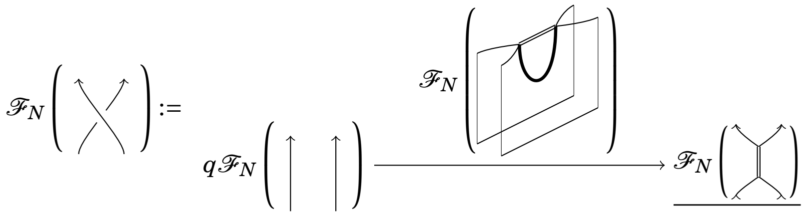
Exercise 3.6. Compute the degree of these foams.

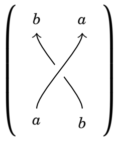
The degree shifts enssure that the differential are homogeneous of degree 0.

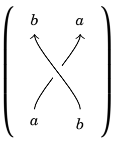
Example 3.7. For $a = b = 1$, one has:

(30) 

and

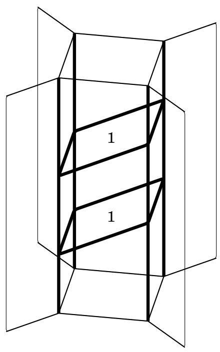
(31) 

Lemma 3.8. *The composition of two consecutive arrows is indeed* 0 *so that* $\mathscr{F}_N$ 

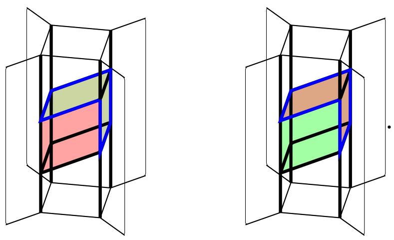
and $\mathscr{F}_N$ 

are indeed chain complexes.

Proof. The composition of two of these foams is easily seen diagrammatically:



Colorings of such foams can be organized in pairs that cancels each other (by looking on how the facets of thickness one in the middle of the square tube are colored):



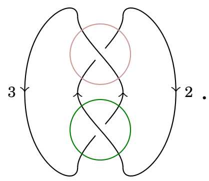
Indeed the only thing which changes in the evaluation of these two colored foams is $s(F, c)$. The blue cycle contribute to θ^+ in one case but not in the other. Hence the sum of these two evaluations is zero. $\square$

3.3. **Link homology.** The construction of link homology starting from $\mathscr{F}_N$ is very similar to that of Khovanov homology. In this section, we'll give idea on how this is extended to colored links.

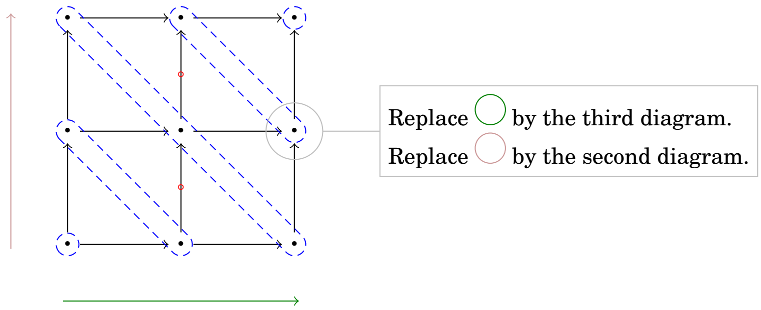
Let D be a knotted MOY diagram. Each crossing of type $(a,b,\pm)$ gives rise locally to $\min(a,b)+1$ diagrams (ordered by homological degree).

We can organize all these diagrams (with formal q-grading shits) in an hyper-rectangle of dimension the number of crossings of D.

Example 3.9. Consider the Hopf link colored by 2 and 3:



Each of the two crossings gives rise to 3 diagrams. Hence we have $3\times 3 = 9$ diagrams to put in an hyper-rectangle:



We can now form a chain complex as follows:

(1) For each diagram apply the functor (and the grading shift);
(2) Put minus signs on arrows of the hyper-rectangle so that there are an odd number of minus signs in every (small) square.
(3) Apply the functor on all diagrams and arrows (and add signs following the previous point).
(4) Take direct sum of all spaces in the same homological degree.
(5) Form a chain complex by considering the sequence of these spaces and maps between them induced by the (signed) arrows.

What we obtain is indeed a chain complex of graded finitely generated free graded R_N-modules: the square of the differential is indeed 0, because of Lemma 3.8 and of the sign choice.

Exercise 3.10. Prove that it is always possible to find such a sign assignment and that different choices of such yields isomorphic chain complexes.

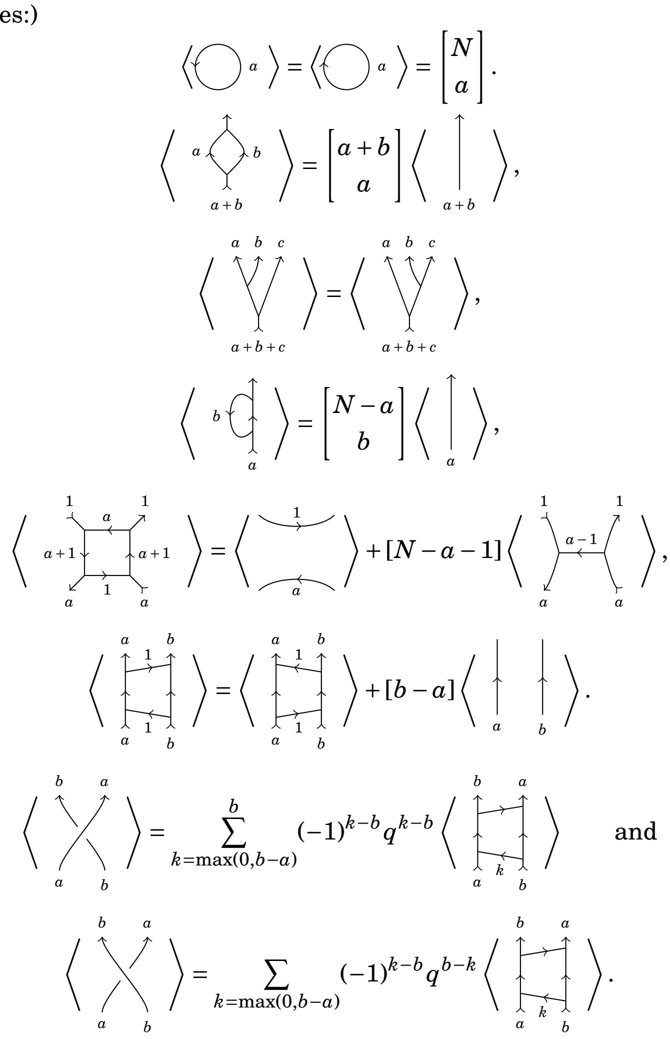
Theorem 3.11. *The homotopy type of this chain complex is an invariant of framed colored oriented link. It categorifies $\langle\cdot\rangle_N$, and can be turned into an unframed colored oriented link invariant by some grading shifts.*

Sketch of the proof. As for the polynomial invariant, one starts with the uncolored case. One need to prove invariance under Reidemeister moves (up to some grading shifts for Reidemeister I). This can be done essentially using the same strategy as Bar-Natan's [BN02] for Khovanov homology. See [Vaz08, ETW18] or [Kho04] for the $\mathfrak{sl}_3$-setting which is close but not completely identical to our setting.

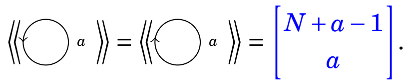
Then one proves invariance under fork-sliding and fork-twisting (up to some grading shifts). See [QR16, ETW18] and finally one deduces from that invariance in the colored setting as for the polynomial invariant. □

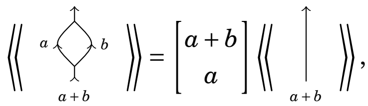
4. Lecture 4: Symmetric $\mathfrak{gl}_N$ homology

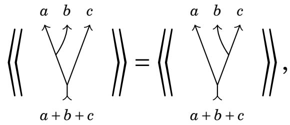
4.1. Symmetric MOY calculus. Recall from Lecture 1 that the colored $\mathfrak{gl}_N$ polynomial framed link invariant can be computed using the following relations (and their mirror images:)

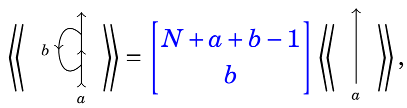


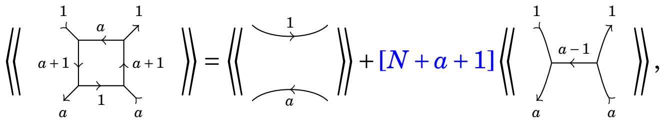
Recall as well, that these identities makes sense for $N < 0$. However some signs appears. We can actually get rid of these signs by looking at the number rotational number of MOY graphs which is a good thing from the categorification perspective. We obtain a new polynomial $\langle\!\langle \cdot \rangle\!\rangle_N$ associated with knot MOY diagrams which satisfies the following identities:

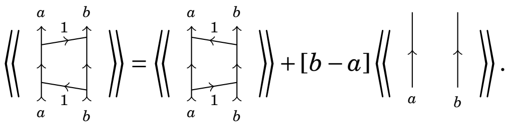
(32) 

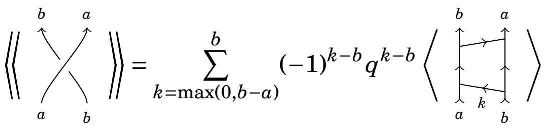
(33) 

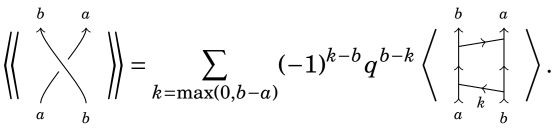
(34) 

(35) 

(36) 

(37) 

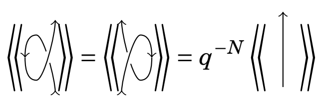
 and

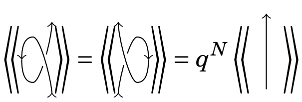


We changed definition of the crossings (or equivalently changed q for q^{-1}). The reason for this change should become clear in a moment.

Proposition 4.1. *The* $\langle\!\langle \cdot \rangle\!\rangle$ *is an invariants of oriented framed colored links.*

The behavior of $\langle\!\langle \cdot \rangle\!\rangle$ regarding the framing is given by:

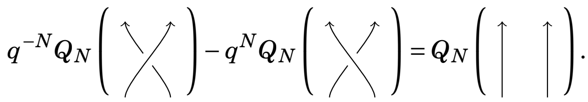
(38) 

(39) 

The exponents of q are the same as in (12) and (13. This is the effect of two different manipulations: on the one hand N is changed to $-N$ on the other hand, q is changed for q^{-1}.

The signs are changed, this is because Reidemeister I move changes the rotational of the diagram by ± 1.

One can construct a framed invariant Q_N from $\langle\!\langle \cdot \rangle\!\rangle_N$ which for uncolored links satisfies:



hence it is equal to P_N. However its colored counterpart of Q_N is different from P_N since the a-colored unknots gives $\left[\begin{smallmatrix} N+a-1 \\ a \end{smallmatrix}\right]$ for the former and $\left[\begin{smallmatrix} N \\ a \end{smallmatrix}\right]$ for the latter.

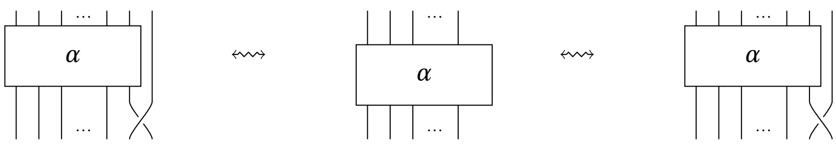
Remark 4.2. The binomial $\binom{N+a-1}{a}$ is the number of ways to choose a elements in $[\![N]\!]$ with possible repetition. It is possible to give a weigthed counterpart of this fact to interpret $\left[\begin{smallmatrix} N+a-1 \\ a \end{smallmatrix}\right]$.

From a representation theoretic point of view $\langle\!\langle \cdot \rangle\!\rangle$ deal with (quantum) symmetric power instead of (quantum) exterior powers.

4.2. **Vinyl graphs and foams.** In order to categorify $\langle\!\langle \cdot \rangle\!\rangle$ we need to restrict to braid closure. This is legitimate because of Alexander and Markov's theorem.

Theorem 4.3 (Alexander and Markov's theorem)**.** *Every colored link is the closure of a colored braid and two colored braids represent the same colored link if and only if one can go from one to the other using the Markov moves I and II:*

Markov I



Markov II

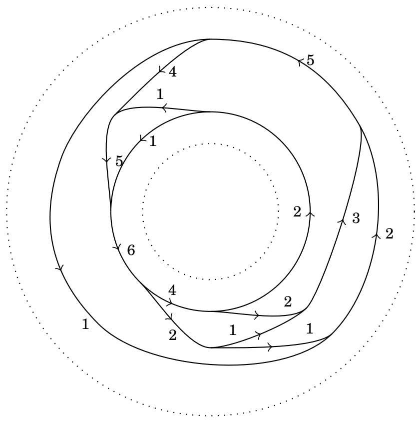
fintely many times.

Braids are oriented upward and are closed on the right.

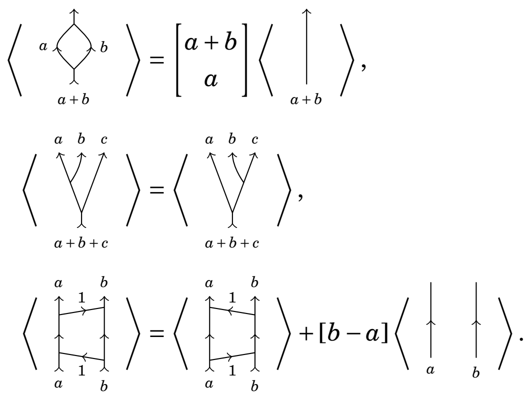
We denote by $\mathscr{A}$ the annulus $\{x \in \mathbb{R}^2 | 1 < \|x\| < 2\}$ and for all $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ in $\mathscr{A}$, we denote by t_x the vector $\left(\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix}\right)$. A *ray* in $\mathbb{R}^2$ is a half-line which starts at O, the origin of $\mathbb{R}^2$.

Definition 4.4. A *vinyl graph* is the image of an abstract closed MOY graph Γ in $\mathscr{A}$ by a smooth embedding such that for every point x in the image of Γ, the tangent vector at this point has a positive scalar product with t_x. The set of vinyl graphs is denoted by $\mathcal{V}$. We define the *level* of a vinyl graph to be the rotational of the underlying MOY graph. If k is a non-negative integer.Vinyl graphs are regarded up to ambient isotopy preserving $\mathscr{A}$.

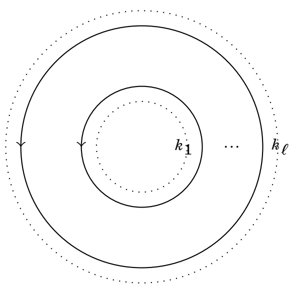
Example 4.5. The following vinyl graph has index 7.



Denote V_k the $\mathbb{Z}[q,q^{-1}]$-module generated by vinyl graphs of index k and modded out by the following relation (and their mirror images):

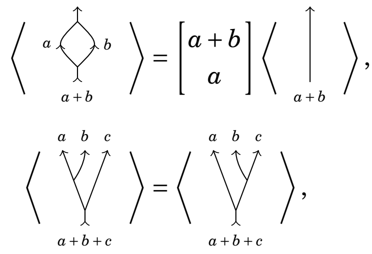


Theorem 4.6 (Queffelec-Rose algorithm [QRS18, QR18])**.** *The $\mathbb{Z}[q,q^{-1}]$-moduel V_k is free with a basis given the $\mathbb{S}_{\underline{k}}$ with $\underline{k} = (k_1,\dots,k_\ell)$, $k_1 \geq k_2 \geq \cdots \geq k_\ell$ and $\sum_{i=1}^{\ell} k_i = k$, where*

$\mathbb{S}_{\underline{k}} =$ 

The single circle $\mathbb{S}_k$ will play a peculiar role in what follows.

Lemma 4.7 (Tree lemma)**.** *Suppose that Γ is an upward oriented MOY graph with boundary from k to $\underline{k}$. Then using only relations:*



Then $\Gamma = P(q)T$ in the appropriate skein module with T a tree and $P(q) \in \mathbb{Z}_{\geq 0}\left[q, q^{-1}\right]$. Moreover in this skein module, all trees are equal.

Sketch of the proof. Induction on the number of merge vertices in the graph. The fact that all trees are equal is trivial. □

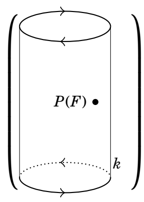
Definition 4.8. A foam F in $\mathscr{A} \times [0,1]$ with vinyl graph boundaries, is *vinyl* if for any point $(x,t) \in F$, the scalar product of the unit normal vector of F at (x,t) with $(x,0)$ is strictly positive. In other words, all oriented sub-surfaces[6] are annuli for which the projection on the last coordinate is a Morse function without critical point.

Vinyl Foams have a well-defined index. For any index k form the non-full subcategory of Foam whose objects are vinyl graphs of index k and morphisms are vinyl foams of index k. It is denoted VFoam_k.

From the Tree Lemma, we obtain:

Lemma 4.9. *Let $F\colon \mathbb{S}_k \to \mathbb{S}_k$ be a vinyl foam. Then there exists a unique symmetric polynomial $P(F) \in \mathbb{Z}[Y_1, \dots Y_k]$, such that*

$$\mathscr{F}_N(F) = \mathscr{F}_N$$



Actually $P(F) = \tau_k(\widehat{F})$, where $\widehat{F}$ is the foam obtained by gluing the two boundary component of F together.

We will now define an evaluation of vinyl $(\mathbb{S}_k, \mathbb{S}_k)$-foams. This will then play a role analogous to τ_N. Remember that N is a fixed positive integer.

Definition 4.10. Let $F\colon \mathbb{S}_k \to \mathbb{S}_k$ be a vinyl foam. The Nth symmetric evaluation of F is the integer $\sigma_N(F)$ defined as the coefficient of $(Y_1 \cdots Y_k)^{N-1}$ in the expansion of $P(F)$ in the monomial basis.

For Γ a vinyl graph, define the graded vector space

$$\mathscr{V}_N(\Gamma) := \bigoplus_{F\colon \mathbb{S}_k \to \Gamma} q^{d_k(F) - k(N-1)}\mathbb{Q}.$$

[6] By subsurface, we mean union of 2-cells with their orientations, which forms a properly embedded oriented surface in $\mathscr{A}$.

and a bilinear form $((\cdot;\cdot))$ on the base given by vinyl foam by:

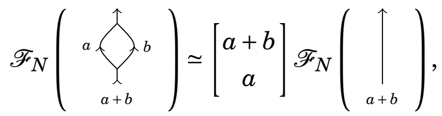
$$((F;G)) := \sigma_N(\overline{F} \circ G).$$

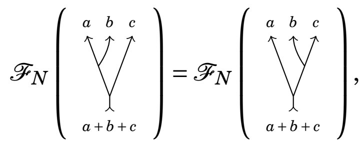
Finally define

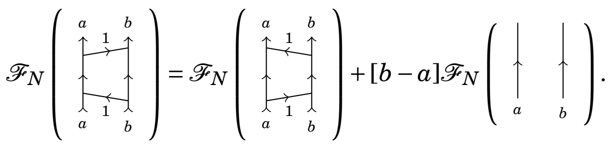
$$\mathscr{S}_N(\Gamma) := \mathscr{V}_N(\Gamma) / \ker((\cdot;\cdot)).$$

As before, $\mathscr{S}_N$ defines a functor from the category of vinyl foam of index k to the category graded vector spaces. Taking direct sum of categories and functors for indexes in $\mathbb{Z}_{\geq 0}$, we get a functor from the category of vinyl foams to the category of graded $\mathbb{Q}$-vector spaces.

Proposition 4.11. *The functor $\mathscr{S}_N$ satisfies the following local relations (and their mirror images):*

(40) 

(41) 

(42) 

and the isomorphisms are realized by images by $\mathscr{S}_N$ of local foams.

Proof. The same foam as for $\mathscr{F}_N$ realizes the same isomorphisms: everything boils down to local relations satisfied by τ_k. □

Proposition 4.12. *The functor $\mathscr{S}_N$ is monoidal (for concentric dijoint union) and we have*

$$\mathscr{F}_N\left(\bigcirc\ k\right) \simeq \begin{bmatrix} N+k-1 \\ k \end{bmatrix} \mathbb{Q}. \tag{43}$$

Idea of proofs. The monoidality comes from the fact that coefficient of $(x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_1+k_2})^{N-1}$ should come from product of coefficient of $(x_1 \cdots x_{k_1})^{N-1}$ and $(x_{k_1+1} \cdots x_{k_1+k_2})^{N-1}$.

A basis of $\mathscr{S}_N(\mathbb{S}_k)$ is given by symmetric monomials "smaller" than $(x_1 \cdots x_k)^{N-1}$ for an appropriate order. □

Corollary 4.13. *The functor $\mathscr{S}_N$ categorifies the symmetric MOY calculus restricted to vinyl graphs.*

Theorem 4.14. *Using the same homological trick to deal with crossing one can extend this categorification to knotted vinyl diagrams and obtain an homological invariant of colored framed links. It can be renormalized to be insensitive to framing.*

Idea of the proof. The fact that we obtain an invariant of colored braid closure is a consequence of that for $\mathscr{F}_N$. Invariance by Markov II (that is Reidemeister I) relies (for the moment) to a connection with triply graded homology and is very difficult.

The problem being that we do not have a categorification of the bad digon relation (35). It would be nice to have a combinatorial proof of invariance □

Interestingly, the homology for $N = 1$ is far from being trivial, although it categorifies the invariant which is constant equal to 1. For $N = 2$, the homology is different from Khovanov homology (and from odd Khovanov homology) although it categorifies the Jones polynomial.

4.3. **Planar approach.** The aim of this last part is to recast what was presented in section 4.2 in a foam free language and using only vinyl graph. This is possible because as stated in next lemma, vinyl foams from $\mathbb{S}_k$ to Γ can be described in a very simple manner. Before stating this lemma, we need a definition:

Definition 4.15. Let $F, G\colon \mathbb{S}_k \to \Gamma$ be two (formal linear combination of) vinyl foams. Then F and G are ∞-*equivalent* if for all $N \in \mathbb{Z}_{\geq 0}$, $\mathscr{F}_N(F) = \mathscr{F}_N(G)$.

Lemma 4.16. *Let $F\colon \mathbb{S}_k \to \Gamma$ be a vinyl foam, then it is ∞-equivalent to a linear combination of tree-like*[7] *vinyl foams with decoration only on the top. Two tree-like foams from $\mathbb{S}_k$ to Γ with same decoration (and only the top) are ∞-equivalent.*

Idea of the proof. The first part is a consequence of the tree lemma and of the fact that $\mathscr{F}_N$ categorifies nicely the digon and (co)associatity relations. □

The previous lemma says that the only relevant information about foams from $\mathbb{S}_k$ to Γ can be stored as a "decoration" of the vinyl graph Γ.

Definition 4.17. A *decoration* $P = (P_e)_{e\in E(\Gamma)}$ is a collection of polynomials indexed by edges of Γ, such that for each edge e, P_e is a symmetric polynomial in $\ell(e)$ variables. The vector space generated by decorations is denoted $\mathscr{D}(\Gamma)$. This space has an algebra structure which comes from the structures of the polynomial algebras used to define it.

Definition 4.18. An omni-coloring of a vinyl graph of index k is a $\mathfrak{gl}_k$-coloring (see Lecture 1) of Γ such that every element of $[\![k]\!]$ is indeed used. In other words, it is a collection of k oriented cycles in Γ which covers all edges with the multiplicity given by the thickness. The set of omnicoloring of Γ is denoted ocol(Γ).

For an omnicoloring c and a decoration P of a vinyl graph Γ, define

$$\sigma_\infty(P,c) = \frac{\prod_{e\in E(\Gamma)} P_e(X_{c(e)})}{\prod_{v\in V_{\mathrm{split}}(V)} \prod_{\substack{i\in c(e_l(v))\\ j\in c(e_r(v))}} (x_i - x_j)}.$$

This definition is designed so that

$$\sigma_\infty(F) = \sum_{c\in \mathrm{ocol}(\Gamma)} \sigma_\infty(P,c)$$

is equal to $\tau_k(F)$ where F is the concatenation of a tree-like foam with decoration on top given by P with a tree-like foam from Γ to $\mathbb{S}_k$ and with the two $\mathbb{S}_k$ glued onto each others.

[7] A *tree-like foam vinyl foam is a vinyl foam for which every vertical slice is a tree.*

Lemma 4.19. *The rational fraction* $\sum_{c\in \mathrm{ocol}(\Gamma)} \sigma_\infty(P,c)$ *is a symmetric polynomial in* k *variables.*

Exercise 4.20. Workout the degree of an homogeneous decoration to match that of the corresponding tree-like foam (one can also look at the evaluation formula to get inspiration).

Finally define $\sigma_N(P)$ to be the coefficient of $(x_1\cdots x_k)^{N-1}$ in the expansion of $\sigma_\infty(P)$ in the monomial basis. We can now endow $\mathscr{D}(\Gamma)$ given by the following bilinear form $(P;Q) = \sigma_N(PQ)$.

Proposition 4.21. *The graded vector space* $\mathscr{D}(\Gamma)/\ker(;)$ *is naturally isomorphic to* $\mathscr{S}_N(\Gamma)$.

Of course with this setup it is not so easy to see morphism between vinyl graphs.

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