A Perspective on
Annular Khovanov Homology

Melissa Zhang (University of Georgia, USA)

Perspectives on Quantum Link Homology Theories

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Roadmap

Lect. 1
Notation, definition, computation
History and Motivation: categorified skein module

Lect. 2
Application: Periodic Links
AKh and Floer theories
Spectral sequences
Knot detection results

Lect. 3
Structure of AKh:
$\mathfrak{sl}_2(\mathbb{C})$ action
Annular Khovanov-Lee Homology
Distilled Numerical Invariants: annular Rasmussen invariants
Annular filtrations on other Khovanov Homologies
Applications to
knot concordance

Lect. 4
transverse knots
braids as MCG ($\mathbb{D}^2$, some points)
Braid detection results

Today's Plan

I. Some topological background

contact structures, transverse knots, braids

Plameneuskaya's invariant ψ

II. A sampling of applications of AKh

(in addition to detection results seen in Lecture 2)

III. Speculations on the role of annular theories in the future: crowdsourcing

AKh, Kh and Contact Topology : Transverse Knots in $\underbrace{(\mathbb{R}^3, \xi_{rot})}_{\text{contact mfld}}$ (or (S^3, ξ_{rot}))

3-mfld, contact structure

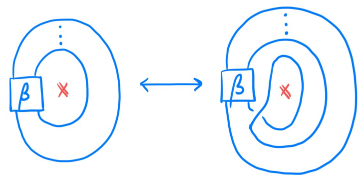
$(\mathbb{R}^3, \xi_{rot})$ Build a maximally nonintegrable! 2-plane field ξ_{rot} in your mind:

1. Take this infinitely long egg noodle with a ½ twist:

And think about its tangent planes.

2. In 3-space, place this string of tangent planes along every line in the xy-plane going through (0,0,0)
3. copy/paste for every z-shift of the xy-plane.

$$\{\text{transverse knots}\} = \frac{\{\text{braid closures}\}}{\text{annular isotopy} + \text{positive markov 2}}:$$



This move is also called positive (de)stabilization.

A knot K in (M, ξ) is transverse if TK_p is transverse to ξ_p at all $p \in K$.

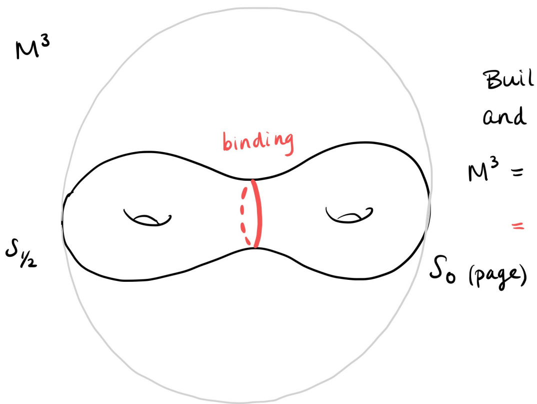
AKh, Kh and Contact Topology: Contact 3-manifolds and open books [Giroux]

A very useful way to represent and work with a contact manifold (M^3, ξ) is to use Giroux's correspondence:

$$\left\{ \begin{matrix} \text{3D contact} \\ \text{manifolds} \end{matrix} \right\} \Big/ \text{isotopy} \longleftrightarrow \left\{ \begin{matrix} \text{Abstract open books} \\ (S, \phi) \end{matrix} \right\} \Big/ \begin{matrix} \text{positive} \\ \text{stabilization} \end{matrix}$$

compact, oriented surface with ∂

$\phi \in MCG(S \text{ rel } \partial)$

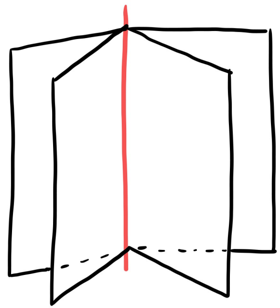


Build $S \times [0,1] / \text{binding}$ and then glue:

$$M^3 = S \times [0,1] \big/ S_0 \sim_\phi S_1$$

$$= M_{S,\phi}$$

eg. (S^3, ξ_{rot})

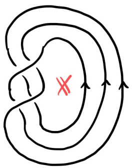


Plamenevskaya's transverse invariant ψ from Kh

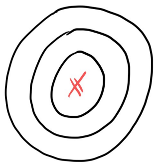
[Plamenevskaya]

For a transverse knot $[\hat{\beta}]$, [Plamenevskaya] defines a cycle $\tilde{\psi} \in Kc(\hat{\beta})$ whose homology class $\psi = [\tilde{\psi}] \in Kh(\hat{\beta})$ is a transverse invariant.

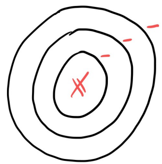
1. Let $\hat{\beta}$ be an annular representative of $[\hat{\beta}]$.



2. There is a resolution consisting only of concentric circles (oriented resolution)



3. $\tilde{\psi}$ is the cycle



The survival / death of ψ in homology can be viewed as an invariant of the transverse knot $[\hat{\beta}]$.

ex.	Prove that $\tilde{\psi}$ is indeed a cycle, i.e. $d_{Kh}(\tilde{\psi}) = 0$.

Open	Is ψ an effective transverse invariant?

"effective" = distinguishes trans. knots better than the data (knot type in S^3) + (self-linking #)

Pause before the sampling begins...

Plamenevskaya's ψ and the contact invariant c from HF

lift of ξ_{st} to $\Sigma(\hat\beta)$

If $\hat\beta$ is a transverse link in (S^3, ξ_{st}), we can build another contact manifold $(\Sigma(\hat\beta), \tilde\xi_{st})$ which is compatible with the natural open-book decomposition of $\Sigma(\hat\beta)$ (S, ϕ) where $\underbrace{\phi \text{ is specified by the braid } \beta}_{\text{Artin generators} \rightsquigarrow \text{Dehn twists on } S}$.

binding is the lift of $\hat\beta$ in $\Sigma(\hat\beta)$

[L. Roberts] $AKh(K) \rightrightarrows \widehat{HFK}(\Sigma(K), \widetilde{axis})$

In this spectral sequence, Plamenevskaya's $\psi(\hat\beta) \in Kh(\hat\beta)$ "corresponds" to Ozsváth-Szabó's contact invariant $c(\tilde\xi_{st}) \in \widehat{HF}(-\Sigma(\hat\beta))$

[Baldwin-Plamenevskaya] generalize this by defining $\underbrace{\widetilde{Kh}(S,\phi)}_{\text{"}\widetilde{Kh}\text{ of an open book"}}$ and $\psi(S,\phi) \in \widetilde{Kh}(S,\phi)$

such that $\psi(S,\phi)$ corresponds to $c(S,\phi) \in \widehat{HF}(-M_{S,\phi})$ under a link-surgeries spectral sequence like the $\underbrace{\text{Ozsváth-Szabó spectral sequence}}_{\widetilde{Kh} \rightrightarrows \widehat{HF}}$.

(use to study tightness and Stein-fillability)

Applications of [Grigsby-Licata-Wehrli]'s annular Rasmussen invariants d_t

[Grigsby-Licata-Wehrli] (and analogously [Truong-Z.] for Sarkar-Seed Szabó homology)

① re-proof of [Plamenevskaya, Shumakovitch]'s s-Bennequin inequality

$$\text{self-linking}(\hat{\beta}) \leq s(\hat{\beta}) - 1$$

($\hat{\beta}$ is a transverse knot)

↑ classical transverse invt

↑ Rasmussen's s

Seifert surface for $\hat{\beta}$

Bennequin's Inequality: $\text{self-linking}(\hat{\beta}) \leq -\chi(\Sigma_{\hat{\beta}})$

② If $\beta \in B_n$ is quasipositive, then $\frac{\partial}{\partial t} d_t(\hat{\beta}) = n$ at all $t \in [0,1)$.

③ If $\frac{\partial}{\partial t}(\hat{\beta}) = n$ for some $t \in [0,1)$, then β is right-veering.

Use d_t to refine

$$QP \subsetneq RV$$

↑ insert more classes here based on their d_t behavior

- Uses [Hubbard-Saltz]'s κ invariant, which measures when ψ dies in the $AKh \rightrightarrows Kh$ spectral sequence, and satisfies

$$\hat{\beta} \ \underline{\text{not}}\ RV \implies \kappa = 2.$$

thesis focuses on AKh!

Applications of [Grigsby-Licata-Wehrli]'s annular Rasmussen invariants d_t

[Martin 19] observes that for fixed n (i.e. B_n), there are a finite number of shapes the function $d_t(\hat{\beta}) : [0,2] \longrightarrow \mathbb{R}$ can have.

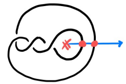
From this, she proves many interesting consequences:

④ Computes d_t (and therefore also s) for <u>all</u> 3-braid closures.

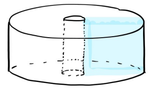
⑤ Death/survival of ψ is entirely determined by s and self-linking #.

She also applies the same technique to Ozsváth-Stipsicz-Szabó's $\Upsilon_K(t)$ from HFK.

Wrapping Conjecture (following [Martin '21])



defn. For $L \subset A \times I$, let $\mathrm{wrap}(L)$ denote the minimal intersection of L with a meridional disk.



Conjecture [Hoste-Przytycki] The Wrapping Conjecture:
Then the maximal non-zero annular degree of the Kauffman skein bracket of L is $\mathrm{wrap}(L)$.

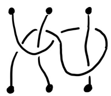
- They exhibit a family of annular links for which this holds.

Recall The graded χ of AKh is the Kauffman skein bracket for A.
[Asaeda-Przytycki-Sikora]

Conjecture [Grigsby] The Categorified Wrapping Conjecture:
The maximal non-zero annular grading of $\mathrm{AKh}(L)$ is $\mathrm{wrap}(L)$.

- [Grigsby-Ni] show this is true for string links.

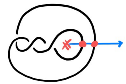
"pure tangle" (nonstandard)

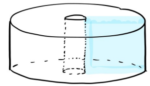


$\{\text{pure braids}\} \subset \{\text{string links}\}$

Wrapping Conjecture

(following [Martin '21])





Conjecture [Grigsby] The Categorified Wrapping Conjecture:
The maximal non-zero annular grading of AKh(L) is wrap(L).

[Martin] has exhibited new infinite families of annular links for which this holds.

- In particular, for these families,
  the max gr_k support of AKh grows unboundedly
  whereas the support of their $\underbrace{\text{Floer-theoretic annular gradings}}_{\text{[Xie] AKh} \rightsquigarrow \text{AHI}}$ are bounded.

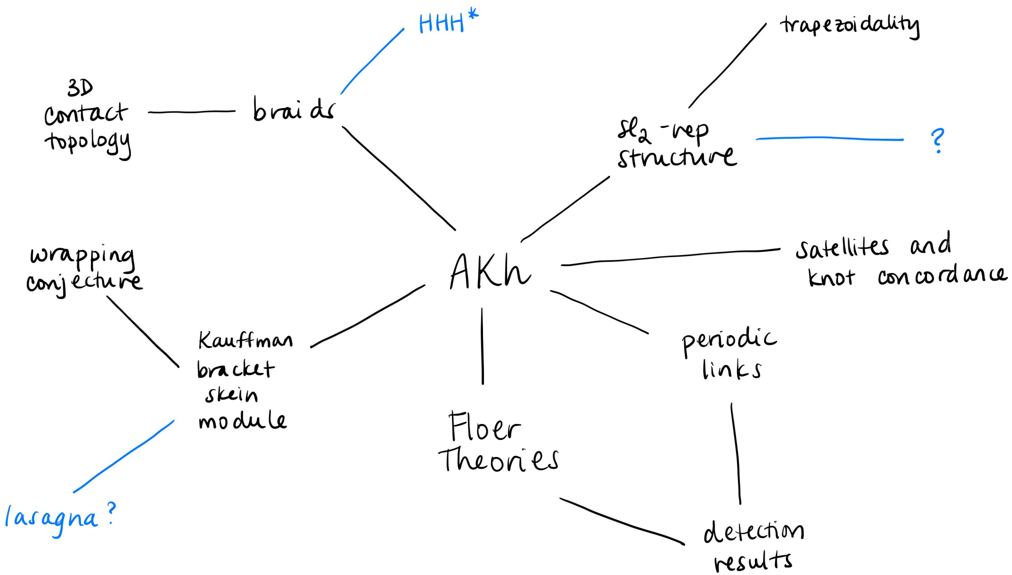
The conjecture can also be viewed as a study of a relationship between

(algebraic/combinatorial) Kauffman skein bracket for A and surfaces in $A \times I \setminus \nu(L)$ (geometric)

[Xie], [Xie-Zhang] (Boyu) AKh(L) is nontrivial in the gr_k of the generalized Thurston norm for all $\underbrace{\text{meridional surfaces}}_{\partial F \text{ is } \amalg \text{ meridians of } L \text{ and maybe some curves in } \partial(A \times I)}$.

Pause before we zoom out and speculate wildly...

Big Picture (of <u>just</u> this lecture series, prototype)



Crowdsourcing:

How do you think AKh (and annular theories in general) will be used in the future?

Thank You !!!