A Perspective On

Annular Khovanov Homology

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Perspectives on Quantum Link Homology Theories

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Roadmap

Lect. 1
- Notation, definition, Computation
- History and Motivation: categorified skein module
- Application: Periodic Links

Lect. 2
- AKh and Floer theories
- Spectral Sequences
- Knot detection results
- Structure of AKh:
  - $sl_2(C)$ action
- Annular Khovanov-Lee Homology
- Distilled Numerical Invariants: annular Rasmussen invariants
- Annular filtrations on other Khovanov Homologies
- Applications to
  - knot concordance
  - transverse knots
    - braids as $\text{MCG} (D^2, \text{some points})$
- Braid detection results

Lect. 3

Lect. 4
Today’s Plan

I. Some topological background
   contact structures, transverse knots, braids
   Plamenevskaya’s invariant $\Psi$

II. A sampling of applications of AKh
    (in addition to detection results seen in Lecture 2)

III. Speculations on the role of annular theories in the future: Crowdsourcing
**AKh, Kh and Contact Topology**: Transverse Knots in $(\mathbb{R}^3, \xi_{\text{rot}})$ (or $(S^3, \xi_{\text{rot}})$)

3-mfd $\gamma$ contact structure

$(\mathbb{R}^3, \xi_{\text{rot}})$ Build a 2-plane field $\xi_{\text{rot}}$ in your mind:

1. Take this infinitely long egg noodle with a $\frac{\pi}{2}$ twist:

   ![Diagram of an egg noodle](image)

   And think about its tangent planes.

2. In 3-space, place this string of tangent planes along every line in the xy-plane going through (0,0,0)

3. Copy/paste for every z-shift of the xy-plane.

A knot $K$ in $(M, \xi)$ is transverse if $TK_p$ is transverse to $\xi_p$ at all $p \in K$.
AKh, Kh and Contact Topology: Contact 3-manifolds and open books [Giroux]

A very useful way to represent and work with a contact manifold \((M^3, \xi)\) is to use Giroux's correspondence:

\[
\begin{align*}
\{ \text{3D Contact manifolds} \} & \leftrightarrow \{ \text{Abstract open books} \} \\
\end{align*}
\]

\(M^3\)

\(S_{1/2}\)

\(S_0\) (page)

Build \(S \times [0, 1] / \text{binding}\) and then glue:

\[M^3 = S \times [0, 1] / S_0 \cong S_1,\]

\[= M_{S, \phi},\phi \in \text{MCG}(S \text{ rel } \partial)\]

\(\phi\) positive stabilization

\(\phi\) \(\in\) \(\text{MCG}(S \text{ rel } \partial)\)

\text{Compact, oriented surface with } \partial

\text{eg.} \ (S^3, \xi_{rot})
Plamenevskaya's transverse invariant \( \Psi \) from \( \text{Kh} \)

For a transverse knot \([\hat{\beta}]\), \cite{Plamenevskaya} defines a cycle \( \hat{\Psi} \in \text{Kh}(\hat{\beta}) \) whose homology class \( \Psi = [\hat{\Psi}] \in \text{Kh}(\hat{\beta}) \) is a transverse invariant.

1. Let \( \hat{\beta} \) be an annular representative of \([\hat{\beta}]\).
2. There is a resolution consisting only of concentric circles (oriented resolution).
3. \( \hat{\Psi} \) is the cycle.

The survival/death of \( \Psi \) in homology can be viewed as an invariant of the transverse knot \([\hat{\beta}]\).

Prove that \( \hat{\Psi} \) is indeed a cycle, i.e. \( d_{\text{Kh}}(\hat{\Psi}) = 0 \).

"Effective" = distinguishes trans. knots better than the data (knot type in \( S^3 \)) + (self-linking #)

Open Is \( \Psi \) an effective transverse invariant?
Pause before the sampling begins...
Plamenevskaya's $\Psi$ and the contact invariant $c$ from $HF$

If $\hat{\beta}$ is a transverse link in $(S^3, \xi_{st})$, we can build another contact manifold $(\Sigma(\beta), \tilde{\xi}_{st})$ which is compatible with the natural open-book decomposition of $\Sigma(\beta) (S, \phi)$ where $\phi$ is specified by the braid $\beta$.

Arnin generators $\mapsto$ Dehn twists on $S$

$L.\, Roberts$ $\quad \text{AKh}(K) \quad \text{HFK} (\Sigma(K), \text{axis})$

In this spectral sequence, Plamenevskaya's $\Psi(\beta) \in \text{Kh}(\beta)$ "corresponds" to Ozsváth-Szabó's contact invariant $c(\xi_{st}) \in \overline{\text{HF}}(-\Sigma(\beta))$

$[\text{Baldwin-Plamenevskaya}]$ generalize this by defining $R\text{h}(S, \phi)$ and $\Psi(S, \phi) \in R\text{h}(S, \phi)$ "$R\text{h}$ of an open book" such that $\Psi(S, \phi)$ corresponds to $c(S, \phi) \in \overline{\text{HF}}(-M_{S, \phi})$

under a link-surgeries spectral sequence like the Ozsváth-Szabó spectral sequence $R\text{h} \Rightarrow \overline{\text{HF}}$

(use to study tightness
and Stein-fillability)
Applications of [Grigsby-Licata-Wehrli]'s annular Rasmussen invariants $d_t$


1. Re-proof of [Plamenevskaya, Shumakovitch]'s $s$-Bennequin inequality

\[ \text{Self-linking}(\hat{\beta}) \leq s(\hat{\beta}) - 1 \quad (\hat{\beta} \text{ is a transverse knot}) \]

\[ \uparrow \text{classical} \quad \uparrow \text{Rasmussen's } s \]

\[ \text{Bennequin's inequality: } \text{self-linking}(\hat{\beta}) \leq -\chi(\Sigma_{\hat{\beta}}) \]

2. If $\beta \in \mathcal{B}_n$ is quasi-positive, then $\frac{\partial}{\partial t} d_t(\hat{\beta}) = n$ at all $t \in [0,1)$.  

3. If $\frac{\partial}{\partial t}(\hat{\beta}) = n$ for some $t \in [0,1)$, then $\beta$ is right-veering.

\[ \{ \text{QP not RV} \} \]

- Uses [Hubbard-Salze]'s $k$ invariant, which measures when $\Psi$
  dies in the $\text{AKh} \Rightarrow \text{Kh}$ spectral sequence, and satisfies
  thesis focuses on AKh!
  \[ \hat{\beta} \text{ not RV } \Rightarrow k = 2. \]
Applications of [Grigsby-Licata-Wehrli]'s annular Rasmussen invariants $d_t$

[Martin 19] observes that for fixed $n$ (i.e. $B_n$), there are a finite number of shapes the function $d_t(\hat{\beta}) : [0, 2] \to \mathbb{R}$ can have.

From this, she proves many interesting consequences:

1. Computes $d_t$ (and therefore also $s$) for all 3-braid closures.

2. Death/survival of $\Psi$ is entirely determined by $s$ and self-linking $\#$.

She also applies the same technique to Ozsváth-Stipsicz-Szabó's $\Xi_K(t)$ from HFK.
Wrapping Conjecture

**defn** For \( L \subset A \times I \), let \( \text{wrap}(L) \) denote the minimal intersection of \( L \) with a meridional disk.

**Conjecture [Hostie-Pretypcki]** The Wrapping Conjecture:

Then the maximal non-zero annular degree of the Kauffman skein bracket of \( L \) is \( \text{wrap}(L) \).

- They exhibit a family of annular links for which this holds.

**Recall** The graded \( \chi \) of \( AKh \) is the Kauffman skein bracket for \( A \).

[Asaeda-Pretypcki-Sikora]

**Conjecture [Grisby]** The Categorified Wrapping Conjecture:

The maximal non-zero annular grading of \( AKh(L) \) is \( \text{wrap}(L) \).

- [Grisby-Ni] show this is true for **string links**

\( \{ \text{pure braids} \} \subset \{ \text{string links} \} \)  

"pure tangle"  

(nonstandard)
Wrapping Conjecture (following [Martin 21])

Conjecture [Gongsby] The Categorified Wrapping Conjecture:
The maximal non-zero annular grading of $\text{AKh}(L)$ is $\text{wrap}(L)$.

[Martin] has exhibited new infinite families of annular links for which this holds.

- In particular, for these families,
  
  the max $g_{rk}$ support of $\text{AKh}$ grows unboundedly
  whereas the support of their Floer-theoretic annular gradings are bounded.

[Xie] $\text{AKh} \cong \text{AH}$

The conjecture can also be viewed as a study of a relationship between

(algebraic/combinatorial) Kauffman bracket for $A$

and surfaces in $A \times I \setminus \nu(L)$ (geometric)

[Xie, Xie-Zhang] $\text{AKh}(L)$ is nontrivial in the $g_{rk}$ of the generalized Thurston norm for all meridional surfaces.

$\mathcal{F}$ is all meridians of $L$ and maybe some curves in $\partial(A \times I)$.
Pause before we zoom out and speculate wildly...
Big Picture  (of just this lecture series, prototype)

- 3D contact topology
- braids
- wrapping conjecture
- Kauffman bracket skein module
- AKh
- Floer Theories
- periodic links
- satellites and knot concordance
- detection results
- HHH*
- trapezoidality
- \( s_{324} \)-rep structure
- lasagna?
Crowdsourcing: How do you think Akh (and annular theories in general) will be used in the future?
Thank You !!!