A Perspective On

Annular Khovanov Homology

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Perspectives on Quantum Link Homology Theories

2021 August 9-13  University of Regensburg
Roadmap

(assuming perfect weather and travel conditions)

Lect. 1
- Notation, definition, Computation
- History and Motivation: categorified skein module
- Application: Periodic Links

Lect. 2
- AKh and Floer theories
- Spectral Sequences
- Knot detection results
- Structure of AKh: $\mathfrak{sl}_2(\mathbb{C})$ action
- Annular Khovanov-Lee Homology
- Distilled Numerical Invariants: annular Rasmussen invariants
- Annular filtrations on other Khovanov Homologies
- Applications to
  - knot concordance
  - transverse knots
  - braids as $\text{MCG}(\mathbb{D}^2, \text{some points})$
  - Braid detection results

Lect. 3

Lect. 4
Structure of $\text{AKh} : \mathfrak{sl}_2(\mathbb{C})$ action

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

- $\mathbb{C}$-basis $\{e, f, h\}$
- Lie bracket $[e, f] = h$
  $[e, h] = -2e$
  $[f, h] = 2f$

Any finite-dimensional representation $\mathcal{U}$ of $\mathfrak{sl}_2(\mathbb{C})$ decomposes (as a $\mathbb{C}$-vector space) into weight-spaces (eigenspaces of $h$):

$$\mathcal{U} = \bigoplus_{\lambda \in \mathbb{C}} \mathcal{U}[\lambda]$$

where $\mathcal{U}[\lambda] = \{v \in \mathcal{U} | h \cdot v = \lambda v\}$

Defining 2-dim'l representation $\mathfrak{sl}_2(\mathbb{C}) \to \text{End}(\mathcal{V})$:

- $h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
- $f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathcal{V} \cong \mathbb{C}^2$

generates maximal torus
raising operator
lowering operator

eq Defining representation:

$$\begin{array}{c}
\bullet \\
\circlearrowleft \\
\text{h} \\
\text{e} \\
\circlearrowleft \\
\text{f} \\
\end{array}$$
Structure of $\text{AKh} : \mathfrak{sl}_2(\mathbb{C})$ action

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

Any finite dimensional representation $\mathcal{U}$ decomposes into irreducible representations.

The irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ are in bijection with $\{0, 1, 2, 3, \ldots\}$ via their highest weight vector.

[Grigsby-Licata-Wehrli]

$\text{AKh}(L; \mathbb{C})$ has an $\mathfrak{sl}_2$ action.

i.e. $\text{AKh}(L; \mathbb{C})$ is an $\mathfrak{sl}_2$ representation.

$\mathfrak{gl}_w$ is the $\mathfrak{sl}_2$ weight-space grading!

$V_0$: \[
\begin{array}{c}
\bullet \\
0
\end{array}
\]

$V_1$: \[
\begin{array}{c}
\text{f} \\
-1 \quad 0 \quad 1
\end{array}
\]

$V_2$: \[
\begin{array}{c}
\text{f} \\
-2 \quad -1 \quad 0 \quad 1 \quad 2
\end{array}
\]

etc.
Structure of $\text{AKh} : \mathfrak{sl}_2(\mathbb{C})$ action

Recall from Lecture 1:

$$
W = \mathbb{C} \langle v^+, v^- \rangle \cong V_1
$$

$$
gr_k(v^+) = +1 \quad gr_k(v^-) = -1
$$

$$
gr_q(v^+) = 1 \quad gr_q(v^-) = 1
$$

$$
W = \mathbb{C} \langle w^+, w^- \rangle \cong V_0 \oplus V_0
$$

$$
gr_k(w^+) = 0 \quad gr_k(w^-) = 0
$$

$$
gr_q(w^+) = \pm 1 \quad gr_q(w^-) = \pm 1
$$

(dual representation to $W$)

$$
W^* = \mathbb{C} \langle v^{*,+}, v^{*,-} \rangle \cong \tilde{W} \cong V_1
$$

$$
gr_k(v^{*,+}) = -1 \quad gr_k(v^{*,-}) = +1
$$

$$
gr_q(v^{*,+}) = -1 \quad gr_q(v^{*,-}) = +1
$$

We need to

1. assign $W$, $W^*$, and $W$ to complete resolutions,
2. define the actions of the operators $e, f, h$, and
3. check that the differential respects the $\mathfrak{sl}_2$-action on chains.
Structure of AKh: $sl_2(C)$ action [Gongsky-Licata-Wehrli]

1. assign $V$, $V^*$, and $W$ to complete resolutions

Define a functor $F: Cob_\phi(A) \rightarrow g\text{Rep}(sl_2)$.

**Example:**

- circle 4 is trivial
- circles 1, 2, 3 are nontrivial
  - circles 1 and 3 enclose an even # of nontrivial
    assign $V$
  - circle 2 encloses an odd # of nontrivial
    assign $V^*$

**BASES**

$V = C \langle v_+, v_- \rangle$

$W = C \langle w_+, w_- \rangle$

$V^* = C \langle v^*_+, v^*_- \rangle$

$\begin{pmatrix}
-1 & +1 \\
-1 & +1
\end{pmatrix}$
Structure of $A_{Kh}: sl_2(C)$ action [Gongsby-Licata-Wehrli]

2. Define the actions of the operators $e$, $f$, and $h$
(We're just choosing a good basis. Since $V$, $V^*$, and $W$ are already $sl_2$ reps, the bracket relations are already satisfied)

Eq. $e$, $f$, and $h$ on $V = C\langle v_+, v_- \rangle$

$e \cdot v_+ = 0, \quad f \cdot v_- = 0$

$e \cdot v_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \cdot v_+$

$f \cdot v_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_+$

$e \cdot v_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_+$

$f \cdot v_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_-$

$W = C\langle w_+, w_- \rangle$

$V^* = C\langle v^*_+, v^*_- \rangle$

$V = C\langle v_+, v_- \rangle$

$W = C\langle w_+, w_- \rangle$

$V^* = C\langle v^*_+, v^*_- \rangle$
Structure of $A Kh: \mathfrak{sl}_2(\mathbb{C})$ action [Glyksby-Licata-Wehrli]

2. define the actions of the operators $e, f, \text{ and } h$
   (We're just choosing a good basis. Since $V, V^*$, and $W$ are already $\mathfrak{sl}_2$ reps, the bracket relations are already satisfied.)

Eqn $e, f, \text{ and } h$ on $V^* = \mathbb{C} \langle v^+_\ast, v^-_\ast \rangle$

\[ v^+_\ast \quad v^-_\ast \]

\[ \begin{array}{c|c|c}
0 & 1 & \end{array} \]

\[ \begin{array}{c}
-1 \\
0 \\
1
\end{array} \]

$e \cdot v^-_\ast = 0, \quad f \cdot v^+_\ast = 0$

$e \cdot v^+_\ast = -(0 \quad 1)^T (1) = (0 \quad 0) = -1 \cdot v^+_\ast$

$e \cdot v^-_\ast = -(0 \quad 1)^T (0) = -(0 \quad 0)(0) = (0 \quad 0) = -v^-_\ast$

$h \cdot v^-_\ast = -(1 \quad 0)^T (0) = (0 \quad -1) \cdot v^+_\ast$

$h \cdot v^+_\ast = -(1 \quad 0)^T (1) = (0 \quad -1)(0) = (0 \quad 0) = -v^+_\ast$

Bases

$V = \mathbb{C} \langle v^+_\ast, v^-_\ast \rangle$

$W = \mathbb{C} \langle w^+_\ast, w^-_\ast \rangle$

$V^* = \mathbb{C} \langle v^+_\ast, v^-_\ast \rangle$
Structure of $\mathfrak{A}K_\mathbb{C}$: $sl_2(\mathbb{C})$ action

3. Check that the differential respects the $sl_2$-action on chains.
   This needs to be checked case-by-case.

Example:

$$V \otimes V^* \rightarrow W$$

Need:

$$e \circ d_{AK} = d_{AK} \circ e$$
$$f \circ d_{AK} = d_{AK} \circ f$$
$$h \circ d_{AK} = d_{AK} \circ h$$

Note: $sl_2 \circ V \otimes V^*$ by a "Leibniz formula":

Example:

$$e \circ (V_+ \otimes V^*)$$
$$= (e \circ V_+) \otimes V^* + V_+ \otimes (e \circ V^*)$$
$$= V_+ \otimes V^* + V_- \otimes (-V^*)$$
$$= V_+ \otimes V^* - V_- \otimes V^*$$

Example:

Confirm that $V \otimes W \xrightarrow{d_{AK}} V$
also commutes with the $sl_2$ action.
Remarks

1. Trapezoidality

Conjecture [Fox 1962]

The coefficients of the Alexander polynomial $\Delta_K(t)$ of an alternating knot $K$ are trapezoidal.

Corollary to $\mathfrak{sl}_2 \mathcal{A}K$ [Grigsby-Licata-Wehrli]

The dimensions of $\mathcal{A}K_{\mathfrak{sl}_2}(L)$ are trapezoidal.

Pf.
- The gr$_K$ support of $\mathcal{A}K(L)$ lies within $\{\text{wrapping number of } D(L) + 2\mathbb{Z}\}$
- $\mathfrak{sl}_2$ irreps are symmetric in weight spaces, centered at $0$.

Open

Investigate whether $\mathcal{A}K \cong \widehat{\mathcal{A}K}$ tells us anything about Fox’s trapezoidality conjecture. (Use $\mathcal{A}K_{\mathfrak{sl}_2}(L) \cong \mathcal{A}K(L)$?)

Recall that $\mathcal{A}K \cong \widehat{\mathcal{A}K}!$
Remarks

2. [Akhmechet-Krishkal-Willis] have described an action of \( e,f,h \) on the annular Khovanov stable homotopy type.

3. There's actually even more structure. [Grigsby-Licata-Wehrli] "Akh and knotted Schur-Weyl representations"

- \( \mathfrak{sl}_2(\Lambda) \) = exterior current algebra of \( \mathfrak{sl}_2 \) acts on \( \text{AKh}(L) \)

- When \( L = \text{m-framed n-cable of } K \subset S^3 \), inside \( A \times I \) (imagine \( K = U, L = \text{torus knot} \)) \( S_n \rtimes \text{AKh}(L) \). The action commutes with the \( \mathfrak{sl}_2(\Lambda) \) action, preserves \( (gr_u, gr_q^r = gr_r, gr_q^l) \) grading, and behaves well under cobordisms.

- So if \( K \subset A \times I \), we may define \( n \)-colored \( \text{AKh} \):

\[
\text{AKh}_n(K) := \text{AKh}(K^n)^{S_n} \subset \text{AKh}(K^n)
\]

\( \downarrow \) framed knot \( \downarrow n \text{ cable} \)
(Pause before we move on to annular Khovanov-Lee homology)
**Annular Khovanov-Lee Homology**

Recall Khovanov-Lee Homology
- chains = $Kc(D)$
- differential = due
  $= d_{\text{akh}} + \Phi_{\text{ue}}$

(The following grading conventions follow those in Rasmussen's $s$-invariant paper)

merge

split

$= d_{0,0} + d_{0,-2} + d_{4,0} + d_{4,2}$

$(gr_k, gr_k)$ bi-degrees of differentials

$= d_{0,0} + d_{0,4} + d_{4,4} + d_{4,0}$

$(gr_k - 2gr_k, gr_k)$ bi-degrees

$gr_k - 2\cdot gr_k$

Also [Grigsby-Licata-Wehrli]!

(filtered version over $F[x]/(x^n)$,
as opposed to graded over $F[x,t]/(x^n-t)$)
Annular Khovanov-Lee Homology

[Grigsby-Licata-Wehrli]

What can we do with this?

1. Study its structure

2. Use the gradings to extract Rasmussen-s-like invariants $d^f$

3. Use $d^f$ to study
   - annular knots (obviously)
   - transverse knots
     $= \frac{\text{braid closures}}{\text{isotopy + Markov 2}}$ \[(\text{Markov 1})\]
   - braid conjugacy classes

\{ details tomorrow \}
Annular Khovanov–Lee Homology  [Grigsby-Licata-Wehrli]

- This is just Lee homology, but with an extra filtration grading $\text{gr}_q - 2 \cdot \text{gr}_k$.

But wait! Looking at the differentials, $\text{gr}_q - t \cdot \text{gr}_k \quad \forall \ t \in [0,2]$ is also a filtration grading!

def The annular concordance invariant $\text{d}_t = \text{Rasmussen's s-invt}$ except where you replace $\text{gr}_q$ with $\text{gr}_t := \text{gr}_q - t \cdot \text{gr}_k$.

- And functionality / behavior under cobordisms is also understood from Kh–Lee homology (eg use Bar-Natan's cobordism category)

<table>
<thead>
<tr>
<th>ex. if you are familiar with Cob/dr</th>
<th>Compute the gr_t filtration degree of the following cobordisms:</th>
</tr>
</thead>
<tbody>
<tr>
<td>annular birth (birth of trivial circle)</td>
<td>(birth of nonnull circle) (saddles are generically annular)</td>
</tr>
</tbody>
</table>
Annular Khovanov-Lee Homology

- One can view this as a strategy for constructing more annular invariants. For example:

\[ d_{s\tilde{s}s} = d_{s\tilde{s}} + \sum_{i=1}^{r} h_{i} \quad \text{(generalized Bar-Natan's perturbation)} \]

We define analogous invariants \( S_{r,t} \) (cf. \( dt \)) and obtain similar applications.

\[ d_{i} = d_{k h} \quad \text{(} h_{i} \text{ is like } \tilde{d}_{k h} \text{, but works over } \mathbb{F}_{2} \text{)} \]

[Truong-Z] For an annular link \( L \subset A \times I \), the Sarkar-Seid-Seabö perturbation of Seabö's geometric spectral sequence admits an annular filtration.

[Akhmechet] has defined \( U(1) \times U(1) \)-equivariant annular Khovanov homology.

cf. [Khovanov-Robert]
Annular Khovanov-Lee Homology: Background for topological applications

Knot Concordance

\[ \mathcal{C} = \mathcal{C}_{\text{smooth}} = \left\{ \text{knots } \subset S^3 \right\} / \text{concordance} \]

(F: \(K_0 \to K_1\), \(F \circ S^3 \times I\))

F "looks" like \(\text{\textbullet}\)

is an abelian group:

- identity = [\(U\)]
  = \{"slice" knots\}
  \(K\) that bound
  \(D^2 \subset B^4\)

- addition = \#

- inverse = mirror

**Example:** Try to "see" the \(D^2\) properly embedded in \(B^4\) with \(\partial D^2 = K \# m(K)\).
**Annular Khovanov-Lee Homology**: Background for topological applications

**Why annular invariants?**

**Knot Concordance**: Satellite operations

\[ P = \text{pattern} \subset A \times I \cong \text{solid torus} \]

\[ K \subset S^3 \quad \text{"companion" knot} \]

\[ \text{nbhd} \left( \begin{array}{c} \text{Satellite knot} \ P(K) : \\ \end{array} \right) \]

**Fact**: If annular knots \( P_1, P_2 \) are annularly concordant, then \( P_1 \cong P_2 \) as morphisms \( P_i : \mathbb{C} \longrightarrow \mathbb{C} \).

\( [K] \longrightarrow [P_i(K)] \)

**Open**: Is the converse true or false? i.e., if \( P_1 \cong P_2 \) as \( \mathbb{C} \longrightarrow \mathbb{C} \), are \( P_1 \) and \( P_2 \) annularly concordant?
Questions and clarifications?