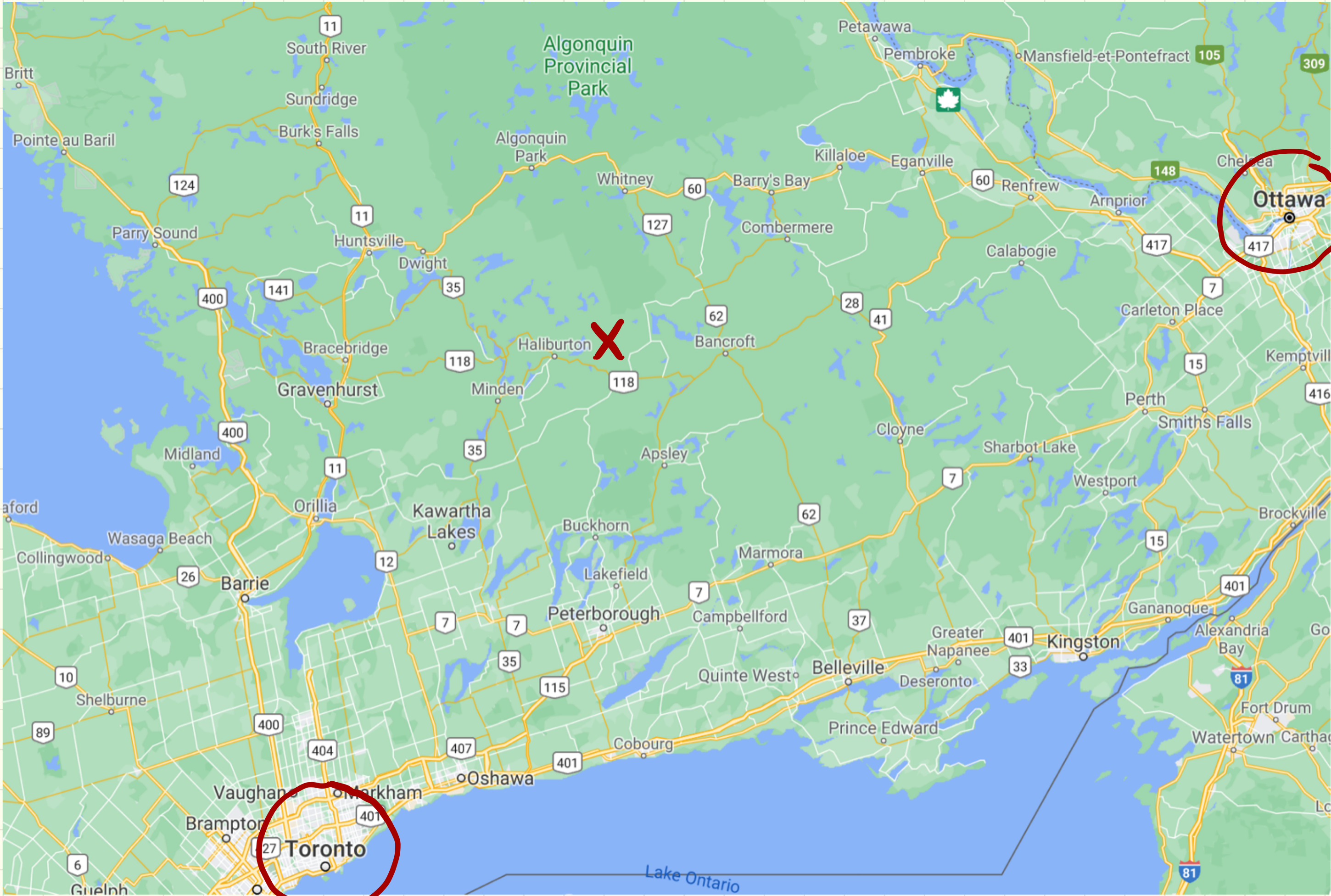
# Heegaard Floer homology and separating tori

(with Jonathan Hanselman and Jake Rasmussen)


Algonquin Provincial Park
Petawawa
Pembroke
Mansfield-et-Pontefract
South River
Britt
Sundridge
Burk's Falls
Pointe au Baril
Algonquin Park
Whitney
Barry's Bay
Killaloe
Eganville
Renfrew
Arnprior
Chelsea
Ottawa
Parry Sound
Huntsville
Dwight
Combermere
Calabogie
Carleton Place
Bracebridge
Haliburton
Bancroft
Kemptville
Gravenhurst
Minden
Perth
Smiths Falls
Midland
Apsley
Cloyne
Sharbot Lake
Westport
Orillia
Kawartha Lakes
Buckhorn
Brockville
Wasaga Beach
Collingwood
Marmora
Barrie
Lakefield
Peterborough
Campbellford
Gananoque
Alexandria Bay
Greater Napanee
Kingston
Shelburne
Quinte West
Belleville
Deseronto
Fort Drum
Prince Edward
Watertown
Cobourg
Oshawa
Vaughan
Markham
Brampton
Toronto
Guelph
Lake Ontario
X

FOR THIS TALK:

- $Y$ IS A CLOSED, CONNECTED, ORIENTABLE 3-MANIFOLD
- $M$ IS A COMPACT, CONNECTED, ORIENTABLE 3-MANIFOLD

$$\partial M \cong S^1 \times S^1$$

- WORK WITH THE 'HAT FLAVOUR' OF HEEGAARD FLOER HOMOLOGY

$$\widehat{HF}(Y)$$

VECTOR SPACE OVER

$$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}.$$

## MAIN THEOREM [HANSELMAN-RASMUSSEN-W.]

> IF $Y$ CONTAINS AN ESSENTIAL SEPARATING TORUS THEN
>
> $$\dim \widehat{HF}(Y) \geq 5$$

**EXP** $\widehat{HF}\Big($ 

$\Big) \cong \mathbb{F}^4$

$S_2$

## FIRST COROLLARY

**SUPPOSE $Y$ IS PRIME. IF $H_1(Y;\mathbb{Q}) = 0$ AND $\dim \widehat{HF}(Y) < 5$ THEN $Y$ IS GEOMETRIC**

<u>PROOF</u> HOMOLOGY RESTRICTION $\Rightarrow$ ANY ESS. TORUS SEPARATES.

SO $\dim < 5$ (THEOREM)

$\Rightarrow$ $Y$ ATOROIDAL.

PERELMAN: ONE OF 7 GEOM. STR.

∎

## SECOND COROLLARY [SEE EFTEKHARY]

INTEGER HOMOLOGY SPHERE L-SPACES ARE ATOROIDAL

PROOF

$\dim \widehat{HF}(Y) = 1$, WHICH IS LESS THAN 5 $\boxtimes$.

THIRD COROLLARY [SEE ARTEM'S TALK]

IF A LINK $L \subset S^3$ CONTAINS AN ESSENTIAL CONWAY SPHERE THEN

$$\dim \widetilde{Kh}(L) \geq 5$$

ARTEM'S GUESS: 17

PROOF AN ESS. CONWAY SPHERE IN $L$ LIFTS TO AN ESS. SEP. TORUS IN $\Sigma_L$ — THE TWO-FOLD BRANCHED COVER OF $S^3$ BR. OVER $L$.

$$5 \leq \dim \widehat{HF}(\Sigma_L) \leq \dim \widetilde{Kh}(L)$$

THEOREM

OZSVÁTH-SZABÓ

$\boxed{\Xi}$

ARTEM: $\widetilde{Kh}(T_1 \cup T_2) \cong HF(\gamma_1^{Kh}, \gamma_2^{BN})$

STRATEGY IS THE SAME:

$Y = M_0 \cup_h M_1$ where $h : \partial M_1 \xrightarrow{\cong} \partial M_0$

~~THE~~ $\widehat{HF}(M)$ CAN BE VIEWED AS A HOMOTOPY CLASS OF AN IMMERSED MULTICURVE IN

$\partial M_0 \setminus \{x\}$ ($x$ A POINT IN $\partial M_0$) TORUS.

IF $\gamma_0 = \widehat{HF}(M_0)$, $\gamma_1 = h(\widehat{HF}(M_1))$ THEN:

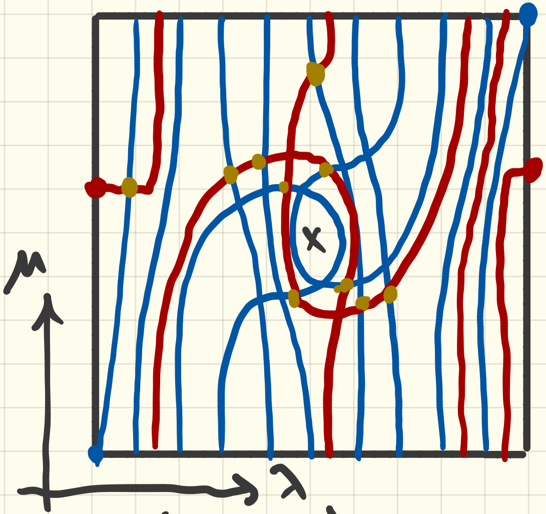
THM [HANSELMAN-RASMUSSEN-W.].

$$\widehat{HF}(Y) \cong HF(\gamma_0, \gamma_1)$$

IN PRACTICE:

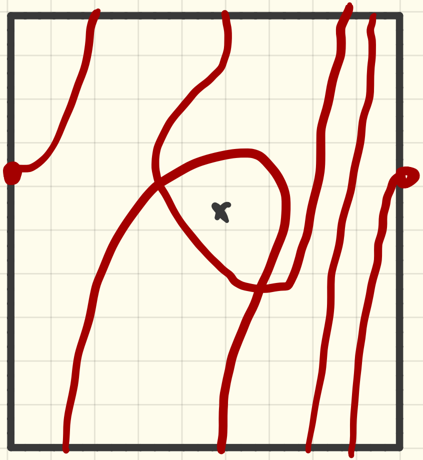
$$\dim HF(\cdot,\cdot) = i(\cdot,\cdot)$$

## EXAMPLE.




$i(\gamma_0, \gamma_1) = 10$

$\widehat{HF}(S^3, T_{3,4})$



$\widehat{HF}(S^3, T_{2,3})$

$h \quad \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

$\lambda \mapsto \lambda + 2\mu$

$\mu \mapsto \mu.$

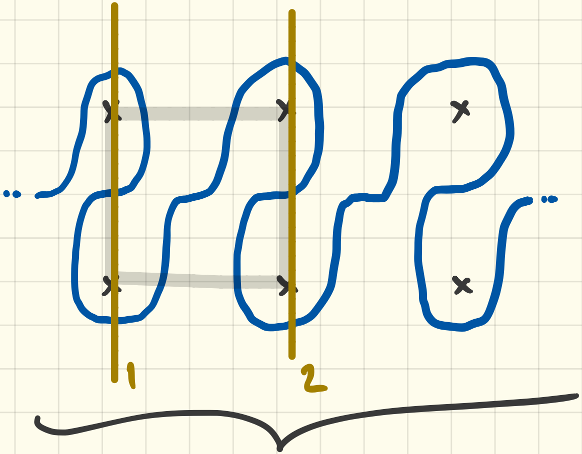
MIGHT CONSIDER $f : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{N} = \mathbb{Z}_{>0}$

$$f(n) = \min_{Y} \left\{ \dim \widehat{HF}(Y) \;\middle|\; Y \text{ CONTAINS } n \text{ DIST. ESS. SEP. TORI} \right\}^{?}$$

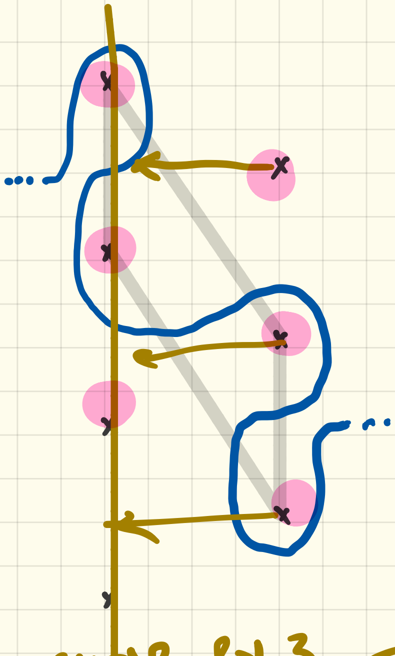
- $f(0) = 1$ SINCE $\widehat{HF}(S^3) \cong \mathbb{F}$
- $f(1) = 5$ MAIN THEOREM
- $f(2) \leq 10$

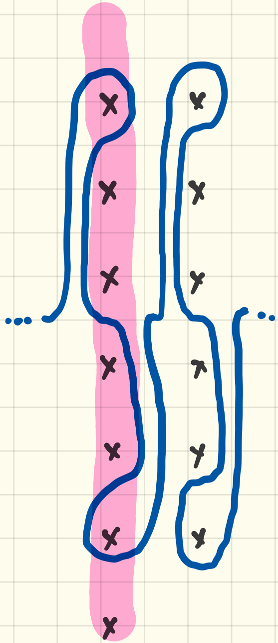
LAST EXAMPLE! NOTE $T_{3,4}$ AND $C_{2,3}(T_{2,3})$ HAVE THE SAME ...... FLOER HOMOLOGY. CALCULATED BY HEDDEN.

FOR CURVES, THERE'S A TRICK! [HANSELMAN-W.]

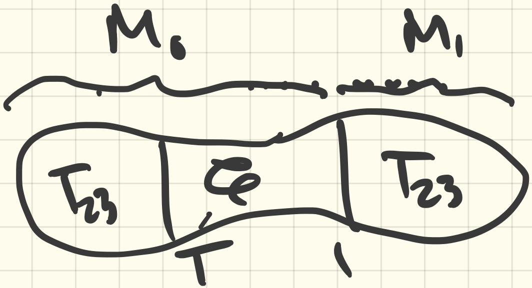



$\widehat{HF}(S^3, T_{2,3})$





2 COPIES → SHEAR BY 3 → VIOLENT CRUSH.



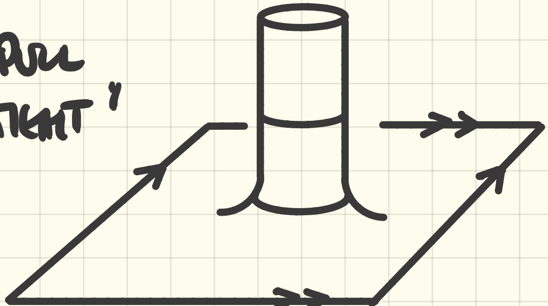

$\widehat{HF}(S^3, \mathcal{C}_{2,3}(T_{2,3}))$

PROBLEM: SAY SOMETHING—ANYTHING— ABOUT $\int$ AS $n$ GROWS.

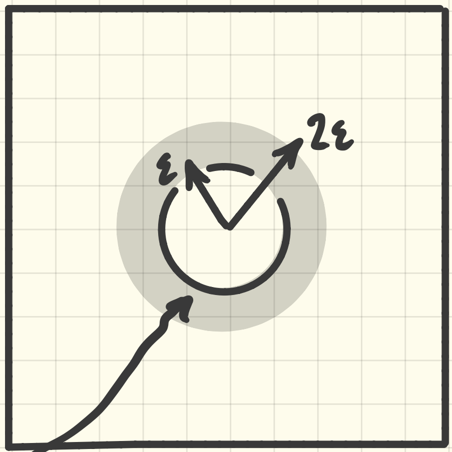
## KEY STEPS IN THE PROOF OF MAIN THM.

GOAL

"PULL $Y_i$ TIGHT"

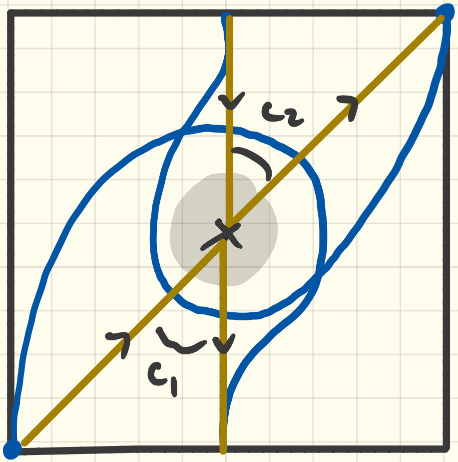


NON-FLAT PART

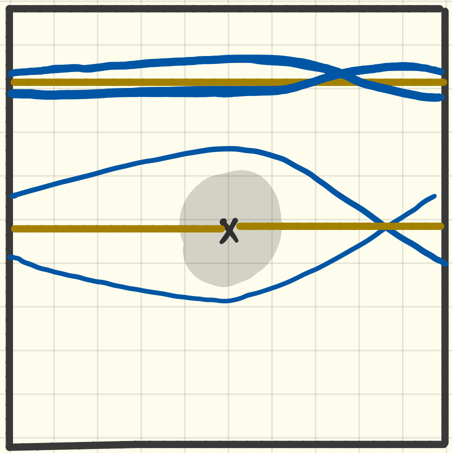



(TAKE $\varepsilon \to 0$)

GEOM. MODEL: NON-POSITIVE CURVATURE.




$T_{2,3}$



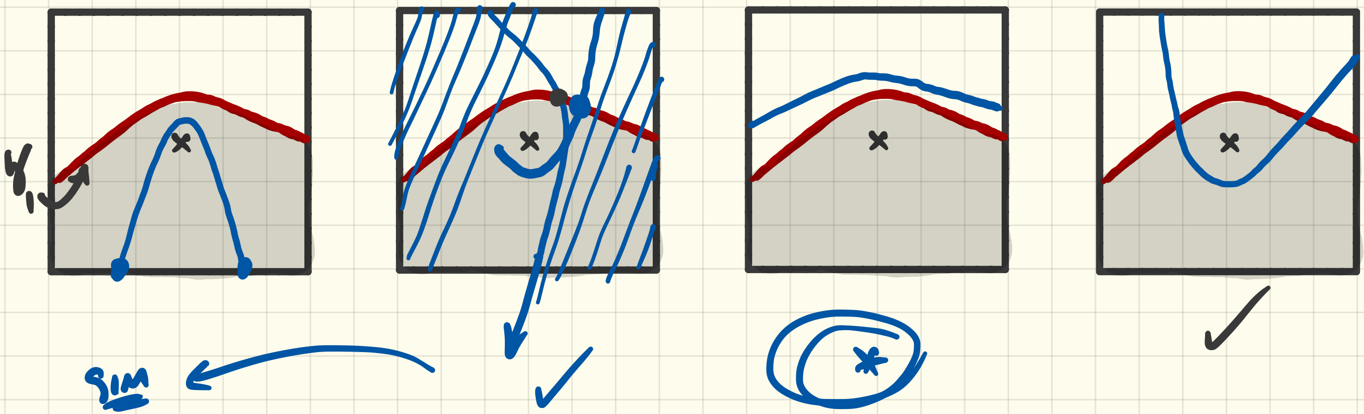
← "STRAIGHT CORNERS"

TWISTED I-BUNDLE OVER KLEIN.

★ INTERESTING CASE: "TRUE CORNERS" ★.

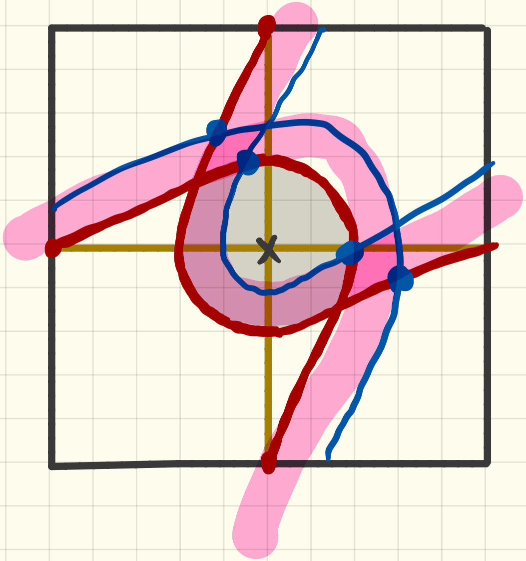
**CLAIM** $i(\gamma_0, \gamma_0) \geq 2 \# \{\text{TRUE CORNERS OF } \gamma_0\}$.

**IDEA** FIX A TRUE CF $\gamma_1$ AND LOOK AT ALL CORNERS OF $\gamma_0$.

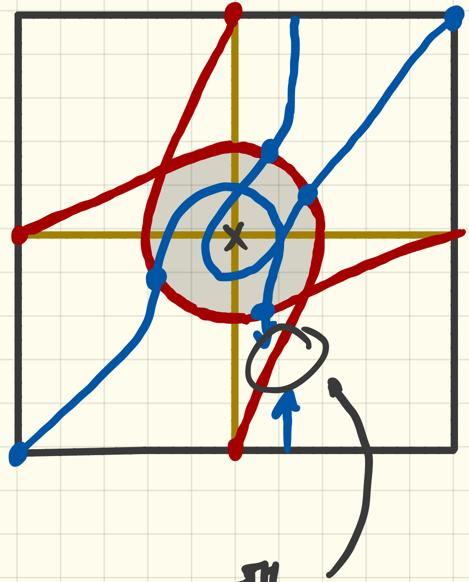



$\blacksquare$.

$\Longrightarrow$ THINK ABOUT $i(\gamma_0, \gamma_0) = 4$
i.e. EXACTLY 2 TRUE CORNERS.



$i(\gamma_0, \gamma_1) = 6$



$\delta^{II}$

$i(\gamma_0, \gamma_1) = 5$

$(M_1 = S^3 \setminus T_{2,3})$