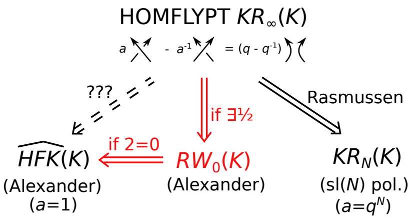
Towards a spectral sequence from HOMFLYPT to HFK

j/w Anna Beliakova, Louis-Hadrien Robert, Emmanuel Wagner

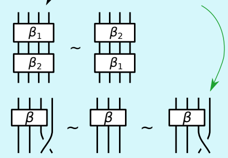
Overview



$\mathbb{Z}$ can be replace by a generic ring
Grading shifts are ignored
I often identify homology with a complex in a htpy category

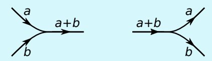
Alexander-Vogel-Markov

$\{\text{links}\} \approx \dfrac{\{\text{braids}\}}{\text{conjugation, stabilization}}$



Webs in $[0,1]\times\mathbb{R}$ or $S^1\times\mathbb{R}$

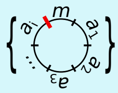
Trivalent graphs with flow condition:



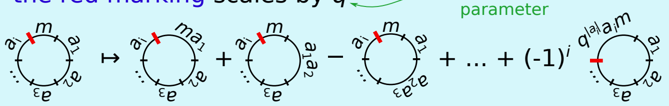
Directed: the projection onto [0,1] or S^1 has no critical points.

Quantum Hochschild

$qCH_i(A, M) :=$



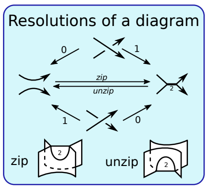
Differential combines arcs together, sliding a through the red marking scales by $q^{|a|}$ (fixed invertible parameter)

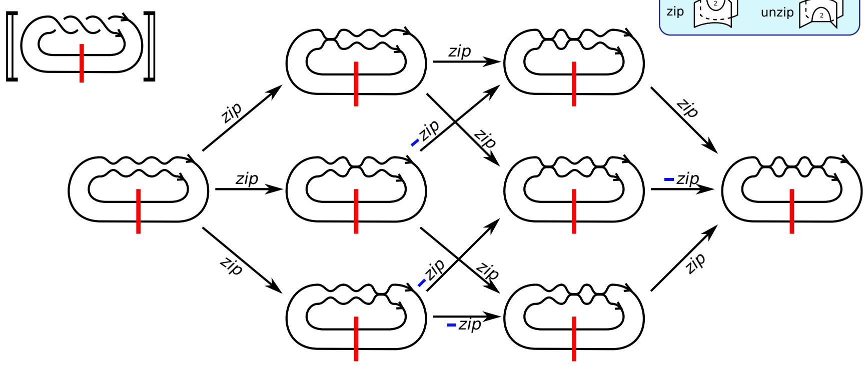


Fact [BPRW]
If $1\text{-}q^d$ is invertible for $d > 0$, then
$$qHH_{>0}(Sym_k, Dec(\omega)) = 0$$

Link homology - general framework

1. Find a braid diagram of a knot
2. Construct the cube of resolutions
3. Sprinkle signs to make squares anticommute
4. Flatten along diagonals





Fact *The htpy type of $[\![\widehat{\beta}]\!]$ is invariant under R2 & R3 moves and conjugation.*

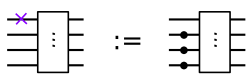
Robert-Wagner homology

[RW] constructed an ∞-evaluation of decorations of an annular web $\widehat{\omega}$ of index k:
$$\langle - \rangle_\infty : Dec(\widehat{\omega}) \to Sym_k$$
It is nondegenerate: $\langle d \rangle_\infty = 0$ iff $d = 0$.

gl(1)-evaluation: $\langle - \rangle_1 := \langle - \rangle_\infty(0,\ldots,0)$
$$RW_1(\widehat{\omega}) = \frac{Dec(\widehat{\omega})}{\ker\langle - \rangle_1}$$

Thm [Robert-Wagner]
The htpy type of the chain complex
$$RW_1(L) := (RW_1)_*[\![\widehat{\beta}]\!]$$
is a (non-trivial) link invariant that categorifies "1".

$RW_0(\widehat{\omega}) \subseteq RW_1(\widehat{\omega})$ is generated by decorations



It is preserved by zips & unzips (and all foams)

Thm [Robert-Wagner]
The htpy type of the chain complex
$$RW_0(K) := (RW_0)_*[\![\widehat{\beta}]\!]$$
is a link invariant that categorifies $\Delta_K(t)$.
If 2 is invertible, then there is a spectral sequence
$$\widetilde{KR}_\infty(K) \Longrightarrow RW_0(K)$$

HOMFLYPT homology

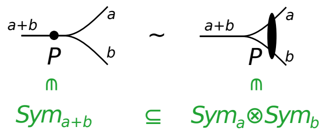
A decoration of a web ω: a collection of symmetric polynomials on edges

a — $P \in Sym_a = \mathbb{Z}[x_1,\ldots,x_a]^{S_a}$

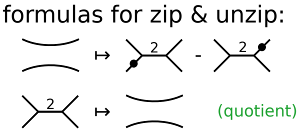
Special case a=1:

1 — n $\equiv$ 1 — x^n

The space of decorations:
$$Dec(\omega) = \frac{\{\text{ decorations of } \omega \text{ }\}}{\text{vertex relation}}$$



Local formulas for zip & unzip:



HOMFLYPT space: (# of endpoints)
$$KR_\infty(\widehat{\omega}) := HH_\bullet(\mathbb{Z}[x_1,\ldots,x_k], Dec(\omega))$$
Note: $HH_0(\ldots) = Dec(\widehat{\omega})$

Thm (Khovanov-Rozansky)
The htpy type of the chain complex
$$KR_\infty(L) := (KR_\infty)_*[\![\widehat{\beta}]\!]$$
is a link invariant that categorifies the HOMFLYPT polynomial.

Pseudo-completion

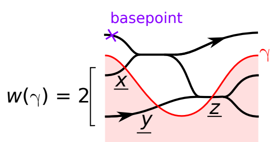
Cannot specialize $Gilm(K)$ at q=1, because 1-q is invertible.
$A(\widehat{\omega})$ can be constructed over $\mathbb{Z}[q^{\pm 1}]$, but it is too large.
A fix: kill what vanishes in the completion!
$$A^P(\widehat{\omega}) := \frac{A(\widehat{\omega}, \mathbb{Z}[q^{\pm 1}])}{\ker(A(\widehat{\omega}, \mathbb{Z}[q^{\pm 1}]) \longrightarrow A(\widehat{\omega}, \mathbb{Z}[q^{\pm 1}]]))}$$
$$P(K) := A^P{}_*[\![\widehat{\beta}]\!]$$

Notes
$P(K)\otimes\mathbb{Z}_2[q^{\pm 1}]] \simeq Gilm(K)\otimes\mathbb{Z}_2 \simeq \widehat{CFK}(K)\otimes\mathbb{Z}_2[q^{\pm 1}]]$
$P(K)$ can be specialized at q=1.

Thm [BPRW] $P(K)|_{q=1} \cong RW_0(K)$. In particular, there is a spectral sequence $\widetilde{KR}_\infty(K) \Longrightarrow P(K)|_{q=1}$ when 2 is invertible.

Gilmore space (revised)

$$A(\widehat{\omega}) = \frac{qHH(\mathbb{Z}[q^{\pm 1}]][x_2,\ldots,x_k], Dec(\omega))}{\text{path relations, dot@base=0}}$$
(power series in q starting at negative powers)



$\text{in}(\gamma) = \prod z_i$
$\text{out}(\gamma) = \prod x_i \prod y_i$
(products of all variables from edges pointing into / out of the red region)

$$\text{in}(\gamma) = q^{w(\gamma)}\text{out}(\gamma)$$

Think of γ as a curve in an annulus that bounds the annular region without the basepoint.

Gilmore complex: $Gilm(K) := A_*[\![\widehat{\beta}]\!]$ (CFK^{-} if no basepoint relations)

Thm [Ozsvath-Szabo]
$Gilm(K)\otimes\mathbb{Z}_2$ is quasi-isomorphic to $\widehat{CFK}(K)\otimes\mathbb{Z}_2[q^{\pm 1}]]$

Rmk The original construction uses another normalization of variables and is defined only for resolutions of a braid diagram.

The spectral sequence $RW_0 \Longrightarrow HFK$ (over $\mathbb{Z}_2$)

We have $\widehat{HFK}(K)\otimes\mathbb{Z}_2[q^{\pm 1}]] \cong Gilm(K) \cong P(K)\otimes\mathbb{Z}_2[q^{\pm 1}]]$, so that all homology have the same rank. Thus

$$\begin{array}{ccc} \widehat{HFK}(K) & \longrightarrow & P(K)|_{q=1} \\ & \searrow^{\cong} & \downarrow qnt \\ & & \text{Free}(P(K))\otimes\mathbb{Z}_2 \end{array}$$

Because $\mathbb{Z}_2[q^{\pm 1}]$ is PID, there is a Bockstein spectral sequence
$$RW_0(K) \cong P(K)|_{q=1} \Longrightarrow \text{Free}(P(K))\otimes\mathbb{Z}_2 \cong \widehat{HFK}(K)$$