Overview

**Robert-Wagner homology**

[RW] constructed an $\infty$-evaluation of decorations of an annular web $\omega$ of index $k$:

\[ \text{Dec}(\omega) \rightarrow \text{Sym}^a \]

It is nondegenerate: \( (d)_a = 0 \) if $d = 0$.

**Thm** [Robert-Wagner]

The htpy type of the chain complex $RW_0(\omega)$ is a (non-trivial) link invariant that categorifies $^a1^n$.

$RW_0(\omega)$ is generated by decorations

\[ = \]

It is preserved by zips & unzips (and all foams)

**Thm** [Robert-Wagner]

The htpy type of the chain complex $RW_0(\omega)$ is a link invariant that categorifies $\Delta_k(t)$. If $2$ is invertible, then there is a spectral sequence $KR_\omega(\omega) \Rightarrow RW_0(\omega)$

**Gilmore space (revised)**

$A(\omega) = \frac{qHH(\text{Dec}(\omega))}{\ker(\text{qbase})}$

path relations, dot@base=0

power series in $q$ starting at negative powers

$w(\gamma) = \text{products of all variables from edges pointing into/out of the red region}$

Think of $\gamma$ as a curve in an annulus that bounds the annular region without the basepoint.

**Gilmore complex**: $Gilm(\omega) := A(\omega)$

$Gilm(\omega) \otimes \mathbb{Z}_2$ is quasi-isomorphic to $\overline{CFK}(\omega) \otimes \mathbb{Z}_2[q^{\pm 1}]$

**Rmk**

The original construction uses another normalization of variables and is defined only for resolutions of a braid diagram.

### Towards a spectral sequence from HOMFLYPT to HFK

**Link homology - general framework**

1. Find a braid diagram of a knot
2. Construct the cube of resolutions
3. Sprinkle signs to make squares anticommute
4. Flatten along diagonals

**Fact** The htpy type of $[\beta]$ is invariant under $R_2$ & $R_3$ moves and conjugation.

**HOMFLYPT homology**

A decoration of a web $\omega$: a collection of symmetric polynomials on edges

\[ aP \in \text{Sym}_a \]

Special case $a=1$:

The space of decorations:

\[ \text{Dec}(\omega) = \{ \text{decorations of } \omega \} \]

vertex relation

\[ \text{Sym}_{a+b} \leq \text{Sym}_a \otimes \text{Sym}_b \]

**Notes**

**Pseudo-completion**

Cannot specialize $Gilm(K)$ at $q=1$, because $1-q$ is invertible.

$A(\omega)$ can be constructed over $\mathbb{Z}[q^{\pm 1}]$, but it is too large.

A fix: kill what vanishes in the completion!

\[ A^p(\omega) := \frac{A(\omega, \mathbb{Z}[q^{\pm 1}])}{\ker(A(\omega, \mathbb{Z}(q^{\pm 1}) \rightarrow A(\omega, \mathbb{Z}[q^{\pm 1}]))} \]

$P(K) := A^p(\beta)$

**Pseudo-completion**

**HOMFLYPT space**: $KR_\omega(\omega) := HH_\omega(\mathbb{Z}[x_1, \ldots, x_n], \text{Dec}(\omega))$

Note: $HH_\omega(\ldots) = \text{Dec}(\omega)$

**Thm** (Khovanov-Rozansky)

The htpy type of the chain complex $KR_\omega(\omega) := (KR_\omega(\omega))$ is a link invariant that categorifies the HOMFLYPT polynomial.

**The spectral sequence $RW_0 \Rightarrow HFK$ (over $\mathbb{Z}_2$)**

We have $\overline{HFK}(\omega) \otimes \mathbb{Z}_2[q^{\pm 1}] \cong Gilm(K) \cong P(K) \otimes \mathbb{Z}_2[q^{\pm 1}]$, so that all homology have the same rank. Thus

\[ \overline{HFK}(\omega) \cong P(K) \otimes \mathbb{Z}_2 \]

Because $\mathbb{Z}_2[q^{\pm 1}]$ is PID, there is a Bockstein spectral sequence $RW_0(\omega) \cong P(K) \otimes \mathbb{Z}_2$. In particular, there is a spectral sequence $KR_\omega(\omega) \Rightarrow P(K)|_{q=1}$ when 2 is invertible.

**Notes**

$P(K) \otimes \mathbb{Z}_2[q^{\pm 1}] = Gilm(K) \otimes \mathbb{Z}_2 = \overline{CFK}(\omega) \otimes \mathbb{Z}_2[q^{\pm 1}]$

$P(K)$ can be specialized at $q=1$.

**Thm** [BPRW] $P(K)_{q=1} \cong RW_0(\omega)$. In particular, there is a spectral sequence $KR_\omega(\omega) \Rightarrow P(K)|_{q=1}$ when 2 is invertible.

**Towards a spectral sequence from HOMFLYPT to HFK**

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