Khovanov:

$$L \subset S^3 \longrightarrow Kh(L) \quad \text{bigraded}$$

$$\chi(Kh(L)) = V_L(q) \quad \text{Jones}$$

$$q^2 V(L_+) - q^{-2} V(L_-) = (q - q^{-1})\, V(L_0)$$

Many Generalizations:

sl_N polynomial:

$$q^N P_N(L_+) - q^{-N} P_N(L_-) = (q - q^{-1}) P(L_0)$$

HOMFLY-PT polynomial

sl_N, minuscule rep'ns $(\Lambda^k V)$

$$H_N(O^{\Lambda^k}) = H^*(Gr(k,N))$$

WRT invts in type A, minuscule colors

Floer:

Heegaard Floer (Ozsváth-Szabó)

$\downarrow$

$$K \subset S^3 \longrightarrow \widehat{HFK}(K) \quad \text{bigraded group}$$

$$\chi(\widehat{HFK}(K)) = \Delta_K(t) \quad \text{Alexander}$$

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\, \Delta(L_0)$$

Link Floer homology (Oz-Sz)

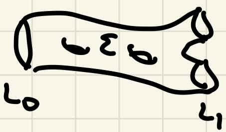
$$L \subset S^3 \to \widehat{HFL}(L) \quad \chi(\widehat{HFL}(L)) = \Delta(L) \prod (1 - t_i)$$

Part of (relative) 3+1 dim'l TQFT

$$K \subset Y \longrightarrow \widehat{HFK}(K)$$

$$\chi(\widehat{HFK}(K)) = (1 - [m])\, \tau_{Tor}(Y - K)$$

Khovanov:



cobordism:

$\Sigma: L_0 \to L_1$ $\qquad$ $\Sigma \subset \mathbb{R}^3 \times I$

induces $F_\Sigma : Kh(L_0) \to Kh(L_1)$

K alternating $\Rightarrow$ $Kh(K)$ determined by $V(K)$, $\sigma(K)$
(E. S. Lee)

$$\dim Kh^r(K) = \det K$$

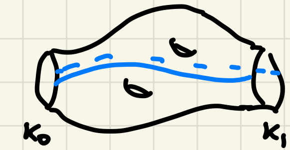
$Kh^r(K) \rightsquigarrow \mathbb{Z}$ (Lee-Turner)

$s(K)$ = filtration of surviving generator

Thm (R): $|s(K)| \le 2g_*(K)$

Floer:

Juhász-Zemke:



$\Sigma: K_1 \to K_2$ decorated cobordism

$\rightsquigarrow F_\Sigma : HFK(K_0) \to HFK(K_1)$

K alternating $\Rightarrow$ $\widehat{HFK}(K)$ determined by $\Delta(K)$, $\sigma(K)$
(Ozsváth-Szabó)

$$\dim \widehat{HFK}(K) = \det(K)$$

$\widehat{HFK}(K) \rightsquigarrow \mathbb{Z}$ (s.s.)

$\tau(K)$ = filtration grading of surviving generator

Thm (Oz-Sz, R): $|\tau(K)| \le g_*(K)$

Guiding Principles:

Specialization:

Polynomial identities

$\Big\downarrow$ categorify

spectral sequences

Ex: $q^2\,\overline{V}(\;) - q^{-2}\,\overline{V}(\;) = (q-q^{-1})\;V(\;)$

$\Rightarrow \overline{V}(K) = \overline{V}(O) = 1$

Lee-Turner: $Kh'(K) \rightsquigarrow \mathbb{Z}$

Unification:

P_N: $\;q^N P_N(\;) - q^{-N} P_N(\;) = (q-q^{-1})P_N(\;)$

all described by HOMFLY-PT:

$$aP(\;) - a^{-1}P(\;) = (q-q^{-1})P(\;)$$

Categorification:

spectral sequences

$\overline{H}(K) \rightsquigarrow \overline{H}_N(K)$

Back to $\widehat{HFK}$:

$$\overline{P}_0(K) = \Delta_K(q^2)$$

Dunfield-Gukov-R
Manolescu ...
Dowlin
Alishahi
Putyra

Is there a s.s.

$\overline{H}(K) \rightsquigarrow \widehat{HFK}(K)$?

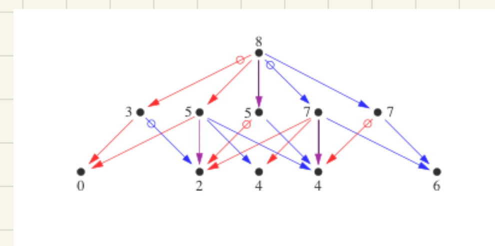
Internal Structure:

Diff'l's "tie together"

generators of $\overline{H}(K)$

c.f. Gukov-Stosic

Eg. for $T(3,4)$



Looking Forward: Extended TQFT's

(Relative) 3+1 d TQFT: is a functor

$$\left\{ \begin{matrix} \text{links in } \mathbb{R}^3 \\ \text{cobordisms} \end{matrix} \right\} \to \left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \mathbb{Z}\text{-linear maps} \end{matrix} \right\}$$

Extended 2+1+1 d TQFT is a 2-functor

$$\left\{ \begin{matrix} \text{Points in } \mathbb{R}^2 \\ \text{tangles in } \mathbb{R}^2 \times I \\ \text{tangle cobordisms} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{categories} \\ \text{functors} \\ \text{natural trans} \end{matrix} \right\}$$

Khovanov: Baked into most definitions of H_N

Floer: Tangle Invariants

Ozsváth - Szabó
Petkova - Vertesi
Zibrowius

Marengon, Willis, Manion, Rouquier

> **Problem:** Put these two sides on an equal footing

Absolute 3+1 dim'l TQFT's:

Khovanov:

Manolescu - Marengon - Sarkar - Willis

K slice in $\overline{\mathbb{C}P}^2 \Rightarrow s(K) \geq 0$

Key ingredient is computation

for $T(n,n)$ = (n,n torus link)

Morrison - Walker - Wedrich

Universal Construction

Key ingredient = functoriality

in $S^3 \times I$

Floer:

functor

$$\left\{ \begin{matrix} \text{3-manifolds} \\ \text{cobordisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{linear maps} \end{matrix} \right\}$$

Extend to 2-functor:

$$\left\{ \begin{matrix} \text{2-manifolds} \\ \text{cobordisms} \\ \text{cobordisms} \\ \text{w/ corners} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{categories} \\ \text{functors} \\ \text{natural trans} \end{matrix} \right\}$$

Lipshitz - Ozsváth - Thurston

Bordered $\widehat{HF}$ (first 2 stages)

$$\boxed{\text{Problem: functoriality for 4-dim'l cobordisms}}$$

Problem: Learn to compute in these categories.

Khovanov: $CKh(D)$ is a chain cx over a category $\mathcal{C}(\partial D)$

Model Thm: $C \in Kom(\mathcal{C}(\partial D))$ is hty ~ to a minimal cx $\overline{C}$.

Hty ~ minimal cxs are $\simeq$.

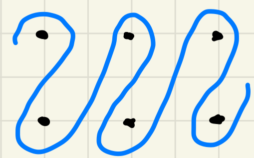
Kotelskiy-Watson-Zibrowius:

Analogous inv't for $Kh(T)$

LOT: If $\partial M = T^2$, $\widehat{CFD}(M)$ is a twisted cx over a category $\mathcal{A}(T^2)$

Hanselman-R-W: Interpret $\widehat{CFD}(M)$ as a multicurve in $T^2 - p$

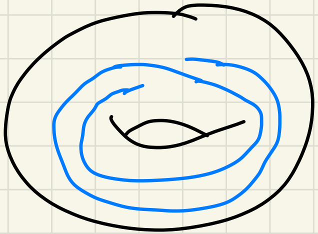
Ex: $M = S^3 - T(2,3)$

$\widehat{HF}(M) =$  / action of $\mathbb{Z}^2$

Zibrowius: tangle $T \subset B^3$, $|\partial T| = 4$

$HFT(T) \in Fuk(S^2 - \partial T)$

Knots in $S^1 \times D^2$:



$K \subset S^1 \times D^2 \longrightarrow Akh(K)$ w/ sl_2 action

$\longrightarrow AH_N(K)$ w/ sl_N action

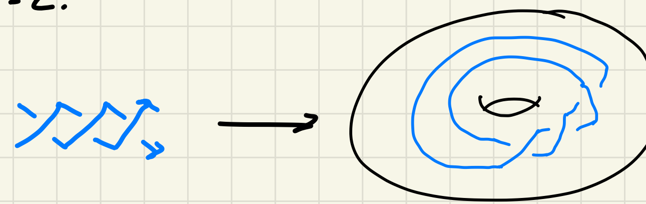
Asaeda-Przytycki-Sikora

Grigsby-Licata-Wehrli

Queffelec-Rose

$\beta \in Br_n \longrightarrow \bar\beta \subset S^1 \times D^2 \longrightarrow \hat\beta \subset S^3$

$n=2$:



$\mathcal{O}(1)$

$\downarrow$

$\mathbb{P}^1 \subset \mathrm{Hilb}^2(\mathbb{C}^2)$

Oblomkov-Rozansky:

1) $\beta \in Br_n \longrightarrow Tr(\beta) \in D^{per}_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^n(\mathbb{C}^2))$

2) $\bar\beta = \bar\beta' \Rightarrow Tr(\beta) = Tr(\beta')$

3) $\overline{H}(\hat\beta) = RHom(\Lambda^\bullet \mathcal{T}^*, Tr(\beta))$

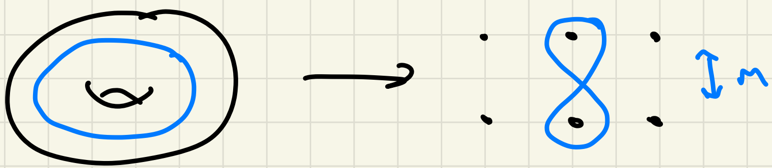
Floer: $Y = S^1 \times D^2 - \nu(K)$

γ_m = meridional suture on $\partial \nu(K)$

$\widehat{CFD}(Y, \gamma_m)$ = twisted cx over $\mathcal{A}(T^2)$

$\rightsquigarrow \widehat{HF}(Y, \gamma_m)$ multicurve in $T^2 - z$.

Ex:

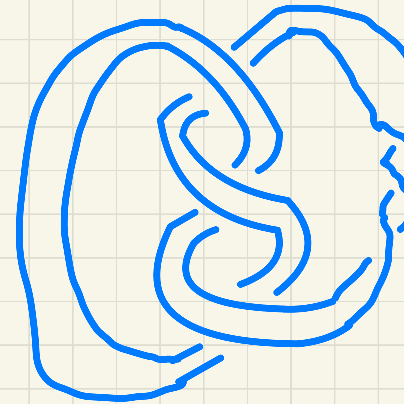


Problem: Relate these 3 notions

E.g. can we derive $AH(\bar{\mathcal{B}})$ from $Tr(\bar{\mathcal{B}})$? What is "spectral sequence"

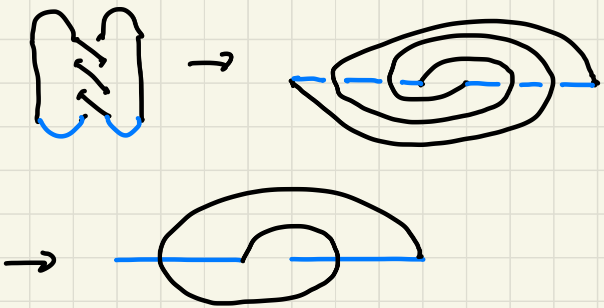
$Tr(\bar{\mathcal{B}}) \rightsquigarrow \widehat{HF}(Y, \gamma_m)$?

Problem: understand $\overline{H}$(satellite knots).

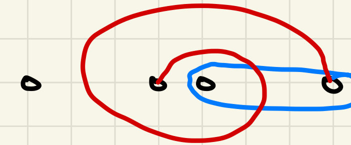


Geometry + Color:

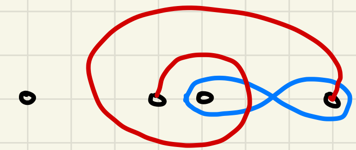
2-bridge knots:



$\widehat{HFK}(T)$ is generated by intersections



Bigelow: $\overline{V}(T)$ counts intersections



In both cases, similar story for k-bridge knots in $\mathrm{Sym}^{k-1}(D^2 - 2k \text{ points}) - \Delta$ (Aganagic)

Generalizations to: P_N (Bigelow)

+ colored Jones (Anghel-Palmer)

Problem: generalize to $P_N^{\Lambda^k}$?

Stabilization in N?

How does $H^{\Lambda^k}(K)$ vary with k?

Knots-Quivers Conjecture: (Kucharski - Reinecke - Stosic - Sulkowski)

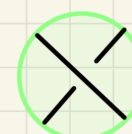
$P^{\Lambda^k}(K)$ for all $k \geq 0$ is explicitly determined by

1) a vectorspace $V(K)$ equipped with
2) 2 linear forms and
3) a quadratic form

For 2-bridge knots, is this data determined by the diagram?

Stosic-Wedrich: Conj holds for **arborescent links** $\longrightarrow$

Built by starting with



and applying