Surface skein algebras, categorification and positivity

joint work with Kevin Walker and Paul Wedrich.

Goal: explore categorifications of $Sk(S)$ and use it to prove positivity conjectures [Fock-Goncharov]

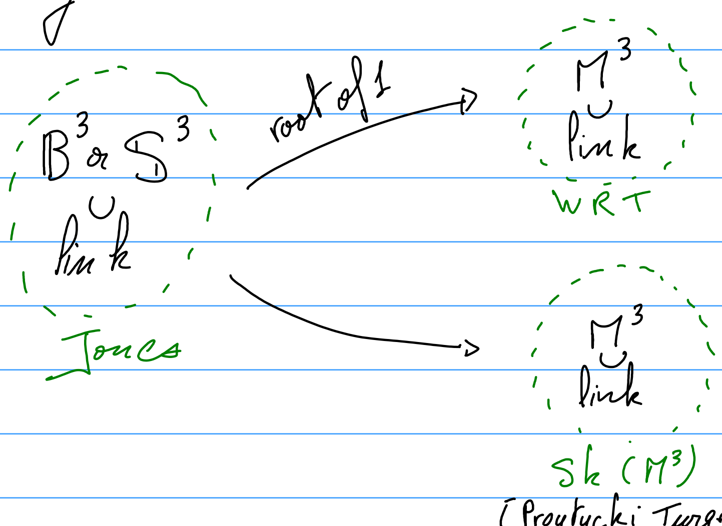
Definition:

$$Sk(S) := \frac{\mathbb{Z}[A^{\pm 1}]\langle \text{links in } S\times I \sim \text{diagrams on } S/\text{isotopy}\rangle}{\langle \times = A\,)(+ A^{-1} \asymp,\ \bigcirc = -A^2 - A^{-2}\rangle}$$

(S: any surface)

\+ structure of an algebra by stacking.

Why should one care about $Sk(S)$?



$$M^3 = H_g \cup_{S_g} H_g$$

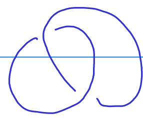
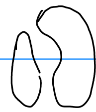
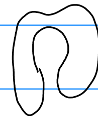
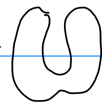
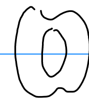
$$\Downarrow$$

$$Sk(M^3) = Sk(H_g) \otimes_{Sk(S_g)} Sk(H_g)$$

algebra structure is central!

Example:

$S = \mathbb{R}^2$

 $\longmapsto A^2$  $+$  $+$  $+ A^{-2}$ 

$$\big(A^2(-A^2-A^{-2})^2 + 2(-A^2-A^{-2}) + A^{-2}(-A^2-A^{-2})^2\big)\varnothing$$
$$= (A^6 + 2A^4 + \cancel{A^2} - \cancel{2A^2} - \cancel{2}A^{-2} + \cancel{A^2} + 2 + \cancel{A}^{-2})\varnothing$$
$$= \underbrace{(A^6 + 2A^4 + 2 - A^{-2})}_{J(\text{Hopf link})}\varnothing$$

$J($  $)$

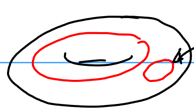
Basis of $Sk(\mathbb{R}^2)$: $\{\varnothing\}$

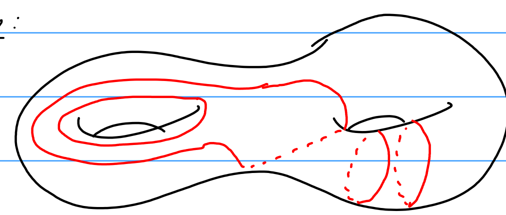
Basis: for S general, a basis is given by multicurves: embedded 1-manifolds, closed, with no inessential loop.

Example:

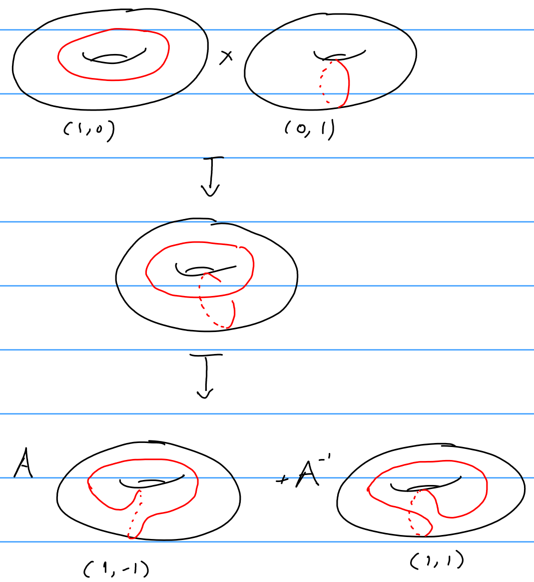
Torus:  $\longleftrightarrow (2,2) = (1,1)^2$

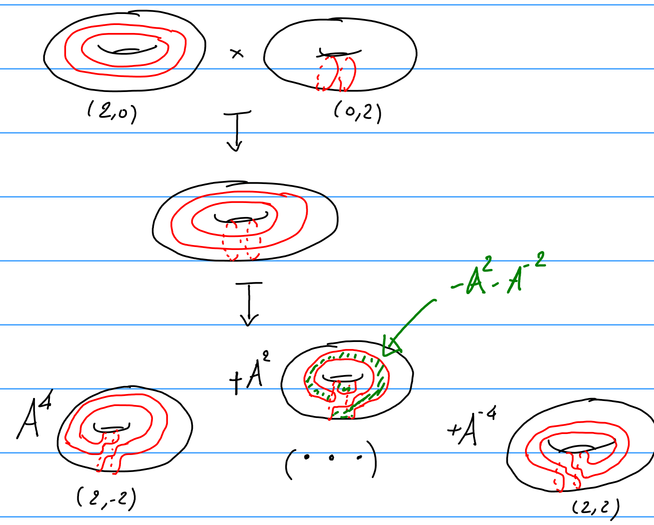
Higher genus:

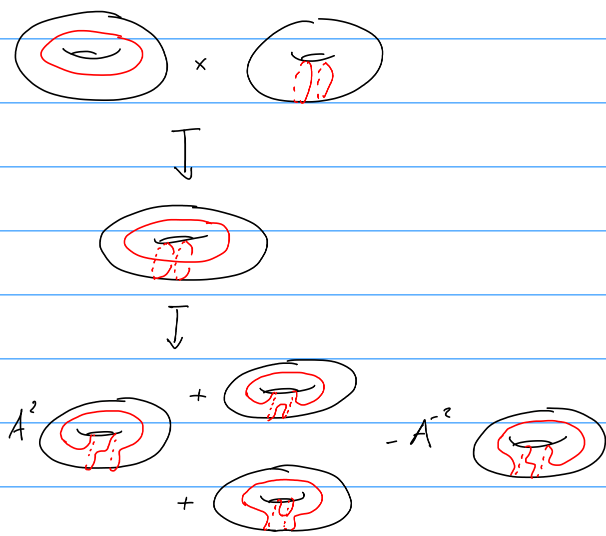
 inessential



Algebra structure: stacking







Chebyshev polynomials of the 1st kind: $T_0 = 2$, $T_1 = X$, $T_k = X T_{k-1} - T_{k-2}$

Theorem [Frohman-Gelca 00]

For $S = \mathbb{T}^2$, the Chebyshev basis $\{\emptyset\} \sqcup \{T_k(m,n),\ m \wedge n = 1,\ k>0\}$ is positive, with explicit formulas:

$$(p,q)_T \times (r,s)_T = A^{\begin{vmatrix} p & r \\ q & s \end{vmatrix}} (p-r, q-s)_T + A^{-\begin{vmatrix} p & r \\ q & s \end{vmatrix}} (p+r, q+s)_T$$

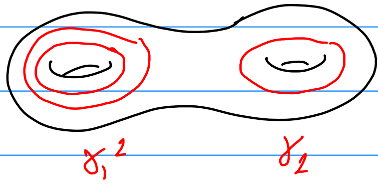
$$(p,q)_T = T_{p\wedge q}\left(\left(\frac{p}{p\wedge q}, \frac{q}{p\wedge q}\right)\right)$$

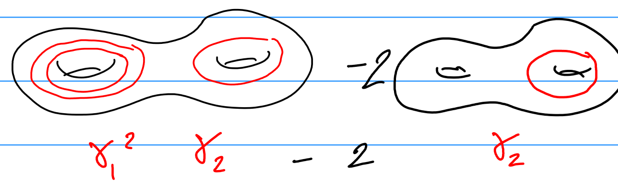
Chebyshev basis

For general S:

multicurve basis $\longleftrightarrow$ Chebyshev basis

$\{\gamma\}$ $\qquad\qquad\qquad$ $\{\gamma_T\}$





Conjecture: [Fock-Goncharov, Thurston, Lê]

$\{\gamma_T\}$ is positive.

[Known for $S = T^2$ and any S if $A=1$ [Thurston]].

Idea: what could categorification tell us about it?

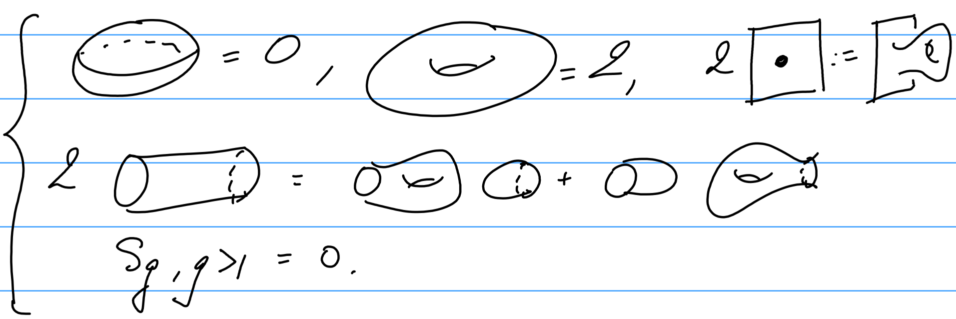
Several steps:

- categorify $Sk(S)$ as a module, and describe the category (for example, is it Krull-Schmidt?)
- categorify the structure of algebra
- lift up the positivity property.

Categorification of $Sk(S)$: Bar-Natan's cobordism category

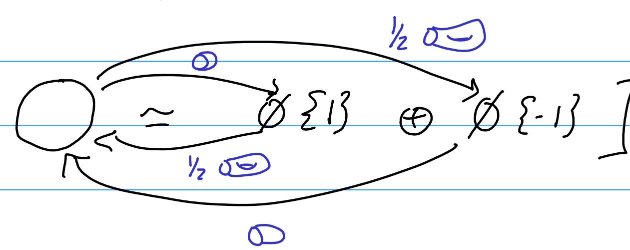
- objects: curves on S with no relations! (and $\oplus$, formal shift)
- morphisms: cobordisms in $S \times [0,1]$ mod relations (matrices of $\mathbb{Q}$-linear combinations)

Relations:



Grading: $-\chi(S)$

Theorem: [Khovanov, Bar-Natan]: $K_0(BN(S)) \simeq Sk(S)$

$[\bigcirc] = A^2 + A^{-2}$ (signs...) $\rightsquigarrow$ 

Trouble: this category is hard to handle...

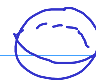
Theorem: [QW18] If $S \neq \mathbb{D}^2$ or $\mathbb{S}^2$, S orientable, then $BN(S)$ is non-negatively graded.

Important because:
- one can hope that $BN(S)^0$ is easier to handle
- $BN(S) \longrightarrow\!\!\!\!\to BN(S)^0$ is a functor.

Defining the grading goes in two steps:

① any object $\simeq \oplus$ essential multicurve $\{\ \}$ $\rightsquigarrow$ $BN(S)^{red}$

② <u>Lemma</u>: if $S \neq S^2$, then a connected surface properly embedded in $S \times [0,1]$ is:

- a disk $\leftarrow$ not in $BN(S)^{red}$
- a sphere bounding a ball $\leftarrow$  $= 0$
- of non-positive χ

$\hookrightarrow$ induces a non-negative grading on $BN(S)^{red}$
$\hookrightarrow$ that propagates in $BN(S)$

<u>Theorem[ish]</u> [QW18]

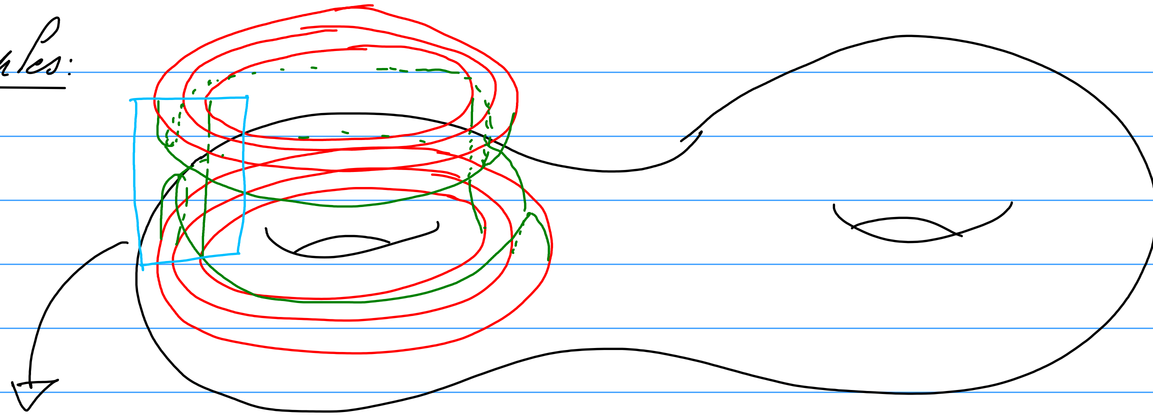
$K \subset S \times [0,1]$

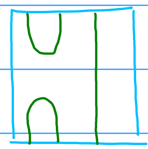
$(BN(K))^0$ is a homotopy invariant that recovers the homology of Asaeda-Przytycki-Sikora

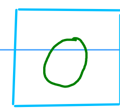
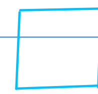
<u>Remark</u>: if $S =$ annulus $\longrightarrow$ sutured annular Khovanov homology [APS, Roberts]

<u>Idea</u>: to capture information specific to S, it's interesting to restrict to $BN(S)^0$.

Simples:

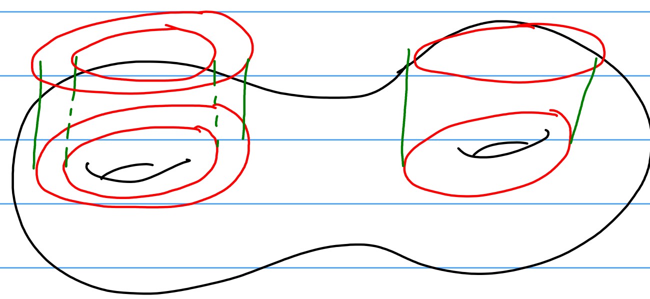
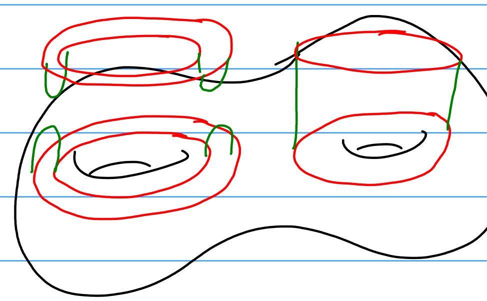


 with relations: • isotopies

•  $= 2$ 

$\Big\}$ TL(2)

Theorem [QW18]: If $S \neq \mathbb{T}^2$ then $BN(S)^\circ$ is semi-simple with simples: unions of $JW \times S^1$ over non-equivalent curves.

Example:  $- \frac{1}{2}$ 

Lemma: The corresponding basis in $Sk(S)$ is:

$\{\gamma_{JW}\}$ made of Chebyshev polynomials of the 2^{nd} kind:

$$U_0 = 1, \quad U_1 = X, \quad U_n = XU_{n-1} - U_{n-2}$$

Conjecture: If $S \neq \mathbb{T}^2$, $\{\gamma_{JW}\}$ is positive.

Theorem [Bousseau 20]: $\{\gamma_T\}$ is positive if $S = \mathbb{T}^2 \setminus \{pt\}$ or $S^2 \setminus 4$ pts.

Nonetheless: to prove the conjecture:

① need ⊗ $\longrightarrow$ takes functorial versions

$\downarrow$

go to $\mathfrak{gl}_2$ foams: full functoriality to be proven!

② find a categorical analog of positivity $\leftrightarrow$ heart of a t-structure

③ prove it's stable under ⊗

① Theorem: [QWW, in progress]:

‖ full list of MM for framed foams

Theorem: $\mathfrak{gl}_2$ Khovanov homology is fully functorial

② just some kind of shifts control on K(SFoam)

③ Theorem: [QWW, in progress]

‖ $\{\gamma_{JW}\}$ is positive

The proof relies on a magic trick:

$$\boxed{A + B \geqslant 0 \implies A \geqslant 0 \text{ and } B \geqslant 0}$$

ex: $\gamma^n = JW_n(\gamma) \oplus JW_{n-2}(\gamma) \oplus \cdots$

$\gamma^n * \gamma' = JW_n(\gamma) * \gamma' \oplus JW_{n-2}(\gamma) * \gamma' \oplus \cdots$ $\mapsto$ enough to prove that $\gamma^n * \gamma' \geqslant 0$

Then assume $\alpha * \beta \not\geqslant 0$ and minimal and use duality.