Coloured Jones and Alexander polynomials unified through Lagrangian intersections in configuration spaces

Outline

Motivation

I. Homological set up

II. Topological model with immersed Lagrangians

III. Topological model with embedded Lagrangians

Motivation

- Aim: Define topological models for $U_q(sl(2))$-quantum invariants

$\boxed{\text{Topological model: graded intersection pairing of homology classes in coverings of configuration spaces}}$

Rasmussen
Dowlin ('18)

Categorifications	Heegaard-Floer Homology	$\xleftarrow{\text{spectral seq}}$	Khovanov Homology Seidel-Smith
Invariants	Alexander polyn $\Delta(K,t)$	$N=2$	Jones polyn. $J(K,q)$
Quantum generalisations (coloured)	$\Phi_N(K,\lambda)$	$N\in\mathbb{N}$	$J_N(K,q)$

Unified topological framework $\xrightarrow{\text{Aim}}$ Categorifications

Bring them together in the same geometric picture

$\boxed{\text{Th}}$ (Bigelow '00, Lawrence)
Noodles and Forks

$\downarrow$ skein rel

$\boxed{\text{Jones polym.}}$

$\boxed{\text{Th 1}}$ (A. '20)
Unified topological model
(immersed Lagrangians)

- uses representation theory

$N \in \mathbb{N}$ colour

$\left(U_q(sl(2)), \; V_N \right)$
q parameter, N-dim repr $\xrightarrow{RT}$ Coloured Jones polym. $J_N(L,q) \in \mathbb{Z}[q^{\pm 1}]$

$\boxed{\text{Th2}}$ (A. '20)
Unified model over 3 variables
(embedded Lagrangians)

- uses Th 1
- suitable for computations

$\left(U_{\xi_N}(sl(2)), \; V_\lambda \right)$
$\xi_N = e^{2\pi i/2N}$, N-dim repr, $\lambda \in \mathbb{C}$ parameter $\longrightarrow$ Coloured Alexander polym. $\Phi_N(L,\lambda) \in \mathbb{Z}[\xi_N^{\pm 1}, \xi_N^{\pm \lambda}]$

$1/\Delta_K$

Unified algebraically
Willetts '20

$\boxed{\text{Representation theory}}$

- Kohno '12
- Ito '15
- Martel '19

$\boxed{\text{Intersections in configuration spaces}}$

$\boxed{\text{Main result}}$ Fix $N \in \mathbb{N}$ - colour of the quantum invariant

Let L - oriented link ; $L = \widehat{\beta_n}$ for $\beta_n \in B_n$

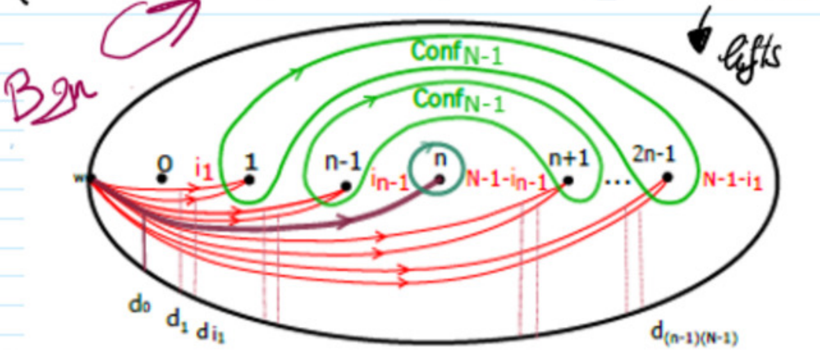
Construction : $\forall \underbrace{i_1, \dots, i_{n-1}}_{\bar{i}} \in \{0, \dots, N-1\} \rightsquigarrow$ Two Lagrangians

$$H_*(\underset{x}{\mathbb{Z}} \oplus \underset{d}{\mathbb{Z}} \text{ covering}) \quad \left(\mathscr{F}_{\bar{i}} \in \mathcal{H}^{-n}_{2n,(n-1)(N-1)+1} ; \mathscr{L}_{\bar{i}} \in \mathcal{H}^{-n,\partial}_{2n,(n-1)(N-1)+1} \right)$$

$\Big\downarrow$ $\parallel$not $\mathcal{H}_{2n,*}$ — lifts

B_{2n} ↻

$\mathrm{Conf}_{(n-1)(N-1)+1}(\mathbb{D}_{2n})$



Def (State sum of Lagrangian intersections)

$$\boxed{\Lambda_N(\beta_n) := u^{-w(\beta_n)} \cdot u^{-(n-1)} \sum_{i_1, \dots, i_{n-1}=0}^{N-1} \langle (\beta_n \cup \mathbb{I}_n) \mathscr{F}_{\bar{i}}, \mathscr{L}_{\bar{i}} \rangle} \in \mathbb{Z}[x^{\pm 1}, d^{\pm 1}, u^{\pm 1}]$$

• $\boxed{\text{Th 2}}$ (A. '20 Unified model through state sums of Lagrangian intersections)

The polynomial in 3 variables Λ_N recovers the N^{th} Coloured Jones and N^{th} Coloured Alexander polynomials for links:

$$\boxed{J_N(L,q) = \Lambda_N(\beta_n) \Big|_{\psi_{1,q,N}}}$$

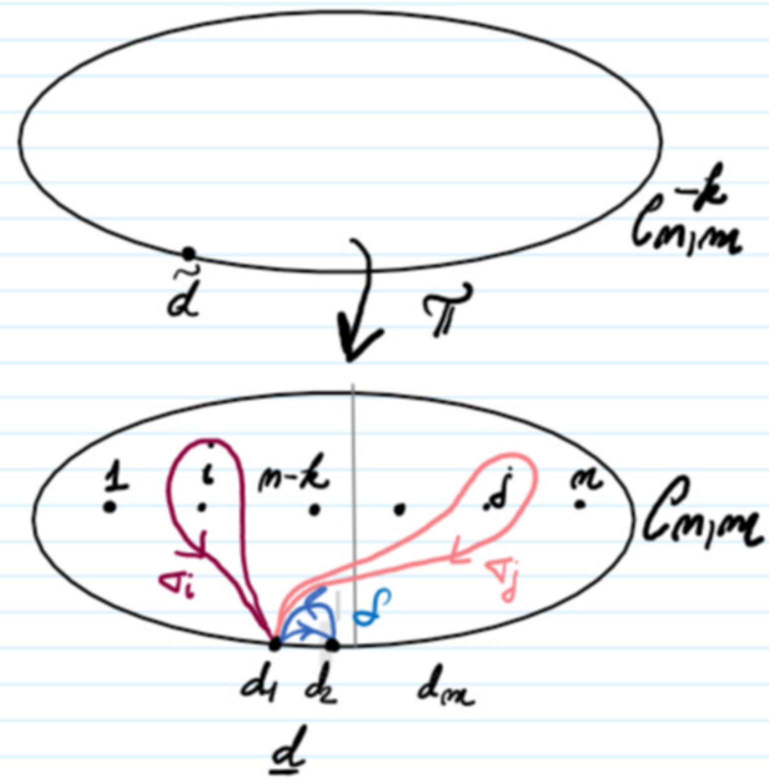
specialisations of coefficients

$$\boxed{\Phi_N(L,\lambda) = \Lambda_N(\beta_n) \Big|_{\psi_{1-N,\xi_N,\lambda}}}$$

I Homological representations

Fix $n, m, k \in \mathbb{N}$

$\mathbb{D}_n := D^2 \setminus \{1, \dots, n\} \rightsquigarrow C_{n,m} = \mathrm{Conf}_m(\mathbb{D}_n)$



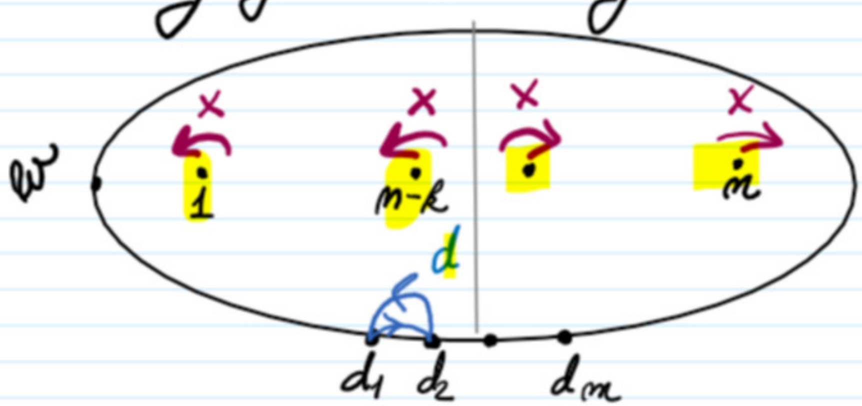
Def: (Local system) Fix $0 \le k \le m$

$$\begin{tikzcd} \pi_1(C_{n,m}) \arrow[r, "ab"] \arrow[rr, bend left, dashed, "\varphi^{-k}"] & \underset{\langle\sigma_i\rangle}{\mathbb{Z}^n} \oplus \underset{\langle\delta\rangle}{\mathbb{Z}} \arrow[r, "aug"] & \underset{\langle x\rangle}{\mathbb{Z}} \oplus \underset{\langle d\rangle}{\mathbb{Z}} \end{tikzcd}$$

$$\langle\sigma_i\rangle \dashrightarrow \begin{cases} x, & 0 \le i \le n-k \\ -x & i > n-k \end{cases}$$

$\rightsquigarrow C^{-k}_{n,m}$ covering sp

Monodromy of the local system φ^{-k}



- Let $w \in \partial D_n$; $\tilde{d} \in \pi^{-1}(\underline{d})$

- $\boxed{\text{Tools}}$: Homology of this covering sp.

① $\mathcal{H}^{-k}_{n,m} \subseteq H^{lf}_m\left(C^{-k}_{n,m}, \pi^{-1}(w); \mathbb{Z}\right)$

$B_n \circlearrowright$ MCG

Borel Moore w.r.t. punctures

colissions

② $\mathcal{H}^{-k,\partial}_{n,m} \subseteq H^{cell}_m\left(C^{-k}_{n,m}, \partial ; \mathbb{Z}\right)$

③ <u>Prop</u> : (A - Palmer) Intersection pairing :

$$\langle , \rangle : \mathcal{H}^{-k}_{n,m} \otimes \mathcal{H}^{-k,\partial}_{n,m} \to \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$$

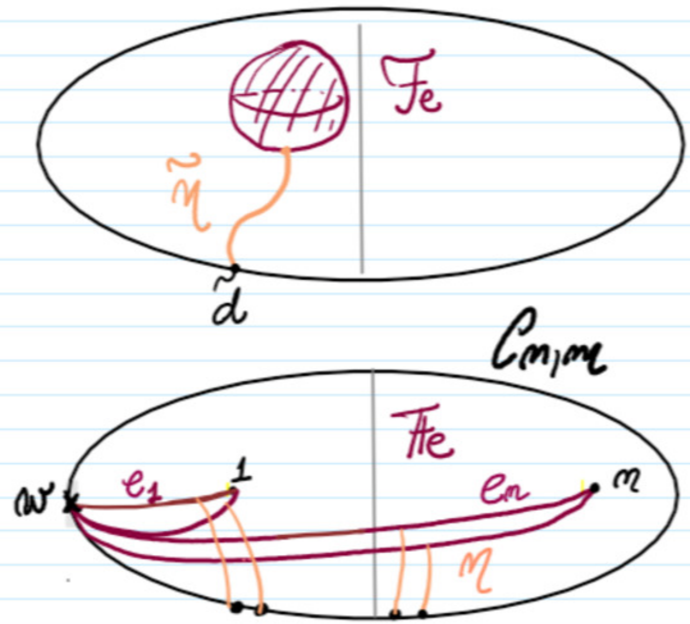
Construction of homology classes

$E_{n,m} := \{ n\text{-partitions of } m \}$

$$\begin{array}{ccc}
\underset{E_{n,m}}{\rotatebox{90}{$\in$}}\; e = (e_1, \dots, e_n) & \rightsquigarrow & \mathcal{F}_e \in \mathcal{H}^{-k}_{n,m} \\
\downarrow \; e_1 + \dots + e_n = m & & \uparrow \text{ lift through } \tilde{\eta}(1) \\
\left(\mathbb{F}_e \subseteq C_{n,m} \;;\; \eta : d \to \mathbb{F}_e \right) & \longrightarrow & \tilde{\eta} \text{ - lift through } \tilde{d}
\end{array}$$

given by the product of curves

path in the config.



$\therefore$ (geometric support ; path to the base point d) $\rightsquigarrow$ homology class $[\tilde{\mathbb{F}}]$

$\mathbb{F}$ $\qquad$ $\eta_{\mathbb{F}}$

- **Prop**: intersection pairing $\langle [\tilde{\mathbb{F}}], [\tilde{\mathbb{G}}] \rangle$ $\overset{\text{encoded}}{\longleftrightarrow}$
  - intersection points $x \in \mathbb{F} \cap \mathbb{G}$
  - graded by the local system via the paths to the base points

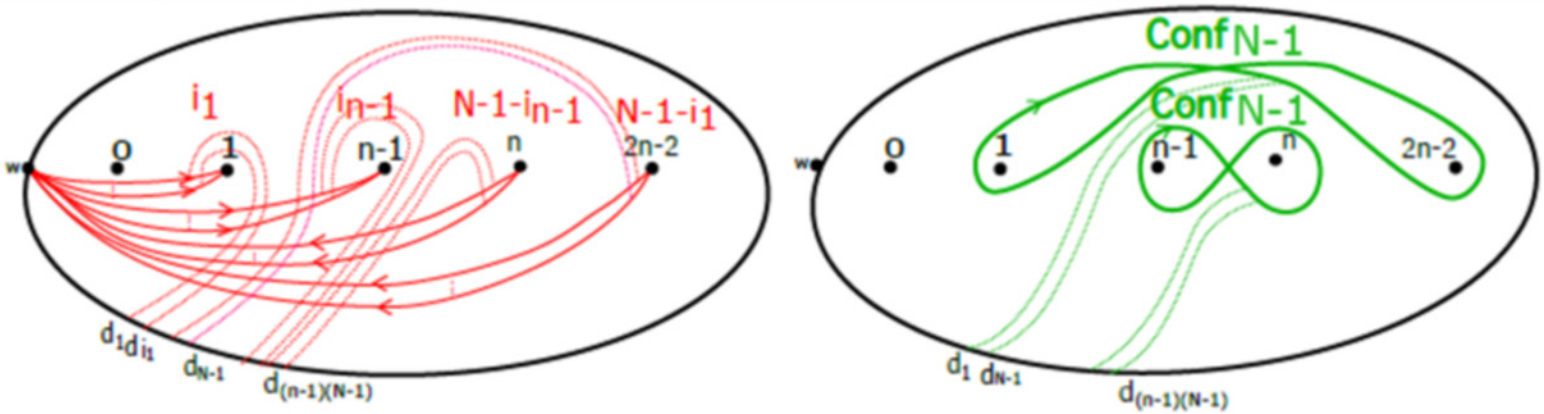
$\boxed{\text{Covering space}}$ $\qquad$ $\boxed{\text{Base configuration space}}$

II Topological model with immersed Lagrangians

Context: L - oriented link ; $L = \hat{\beta}_n$ $\beta_n \in B_n$; Fix $N \in \mathbb{N}$

• **Def** (Main classes) $\underline{i} = (i_1, \dots, i_{n-1}) : i_j \in \{0, \dots, N-1\}$

$$\tilde{\mathcal{U}}_{\underline{i}} \in \mathcal{H}^{\circ}_{2n-1,(n-1)(N-1)} \qquad \mathcal{G}^N_n \in \mathcal{H}^{\partial,\partial}_{2n-1,(n-1)(N-1)}$$



• **Def** : (Homology Classes)

$$\left(\mathcal{E}^N_n := \sum_{\underline{i}=0}^{N-1} d^{-\sum i_k} \cdot \tilde{\mathcal{U}}_{\underline{i}} , \ \mathcal{G}^N_n \right)$$

• **Not** (Specialisation of coefficients) Let $c \in \mathbb{Z}$

$$\psi_{(c),q,\lambda} : \mathbb{Z}[u^{\pm 1}, x^{\pm 1}, d^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}, q^{\pm \lambda}] \qquad \begin{cases} u \rightsquigarrow q^{\lambda} \\ x \rightsquigarrow q^{2\lambda} \\ d \rightsquigarrow q^{-2} \end{cases}$$

$\psi_{q,\lambda}$

[

- $\boxed{\text{Th 1}}$ (A. '20) Topological model via immersed Lagrangians) $L = \widehat{\beta_n}$

Let $\boxed{I_N(\beta_n) := \langle (\beta_n \cup \mathbb{1}_{n-1})\, \mathcal{E}_n^N, \mathcal{G}_n^N \rangle} \in \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$

Then, I_N recovers the N^{th} col. Jones and N^{th} col. Alexander polyn.:

$$\boxed{J_N(L, q) = q^{-(N-1)\, w(\beta_n)} \cdot q^{-(n-1)(N-1)} \cdot I_N(\beta_n) \Big|_{\psi_{q, N-1}}}$$

Not ψ_J

$$\boxed{\Phi_N(L, \lambda) = \xi_N^{(N-1)\cdot\lambda\cdot w(\beta_n)} \cdot \xi_N^{(N-1)(n-1)\lambda} \cdot I_N(\beta_n) \Big|_{\psi_{\xi_N, \lambda}}}$$

ψ_Φ

]

Construction and idea of proof

$\boxed{\text{Algebraic context}}$ $\quad (U_q(sl(2)), R)$ over $\mathbb{Z}[q^{\pm 1}, s^{\pm 1}]$

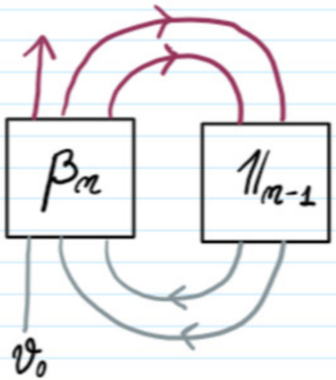
$N \in \mathbb{N}$ $\qquad \hat{\mathbb{V}} := \langle v_0, v_1, \dots, v_{N-1}, v_N, \dots \rangle$ Verma module

Specialisations of variables

$$\begin{tikzcd}
 & \hat{\mathbb{V}} \\
 & \mathbb{V}_N \arrow[u, phantom, "\cup"] \arrow[dl, dashed, "s \to q^{N-1}"'] \arrow[dr, dashed, "{s \to \xi_N^\lambda,\ q \to \xi_N}"] & \\
V_N \arrow[d, squiggly, "\psi_J"'] & & V_\lambda \arrow[d, squiggly, "\psi_\Phi"] \\
J_N & & \Phi_N
\end{tikzcd}$$

Step 1 — Definition of $\mathcal{J}_N$ and Φ_N in this set up $\quad L = \hat{\beta}_n$

Diagramatically



$ev_{\mathcal{J}}$: $\curvearrowright$ $\quad v_i \otimes v_{N-1-i} \mapsto s q^{-2i}$ $\quad$ otherwise 0 $\rightsquigarrow \mathcal{J}_N$

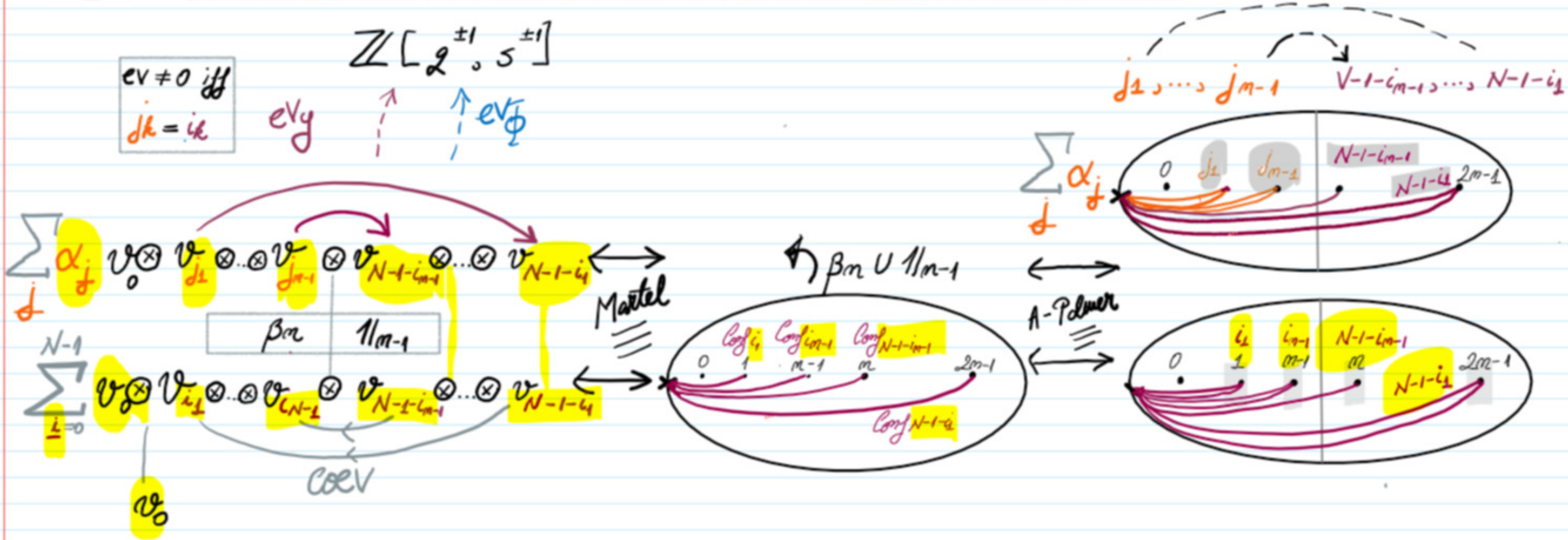
ev_{Φ} : $\curvearrowright$ $\quad v_i \otimes v_{N-1-i} \mapsto s^{1-N} q^{-2i}$ $\quad$ otherwise 0 $\rightsquigarrow \Phi_N$

coev : $\cup$ $\quad 1 \mapsto \sum_i v_i \otimes v_{N-1-i}$

See both invariants from a construction over 2-varibles

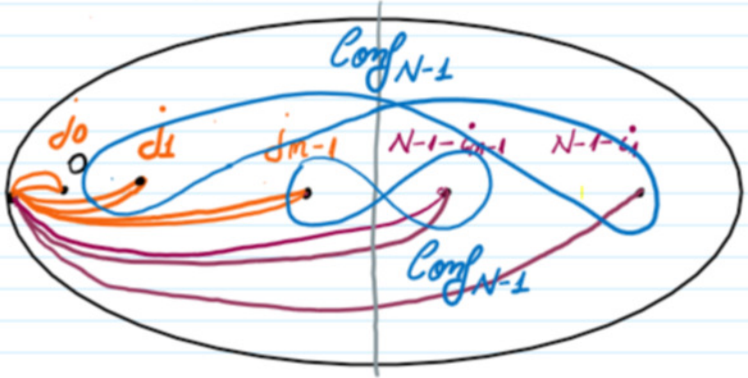
Extend $ev_{\mathcal{J}}$ and ev_{Φ} on all vectors from the Verma module with zero unless they are from $\mathbb{V}_N$

Step 2 Use the weight spaces from the Verma module



Step 3 | Construction of the dual class

We need a dual class which intersects $\mathscr{F}_{j_0, j_1, \ldots, j_{m-1}, N-1-i_{m-1}, \ldots, N-1-i_1}$ non-zero iff $\boxed{\begin{array}{l} j_0 = 0 \\ j_k = i_k \end{array}}$



$\mathscr{G}_m^N$

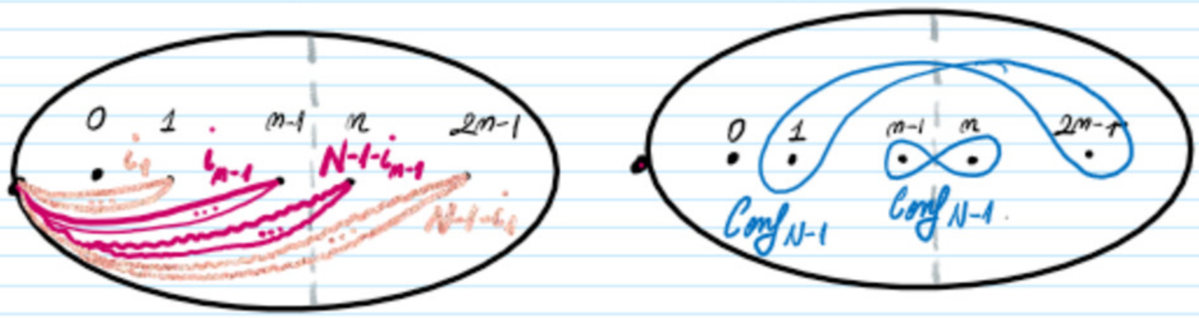
- $\boxed{\text{Corollary 1}}$ (Relation to Bigelow's model for the Jones polynomial)

  Th 1 for $N=2$ recovers Bigelow's classes:

  $\mathcal{F}_m^2 \to$ forks $\mathcal{G}_m^2 \to$ noodles

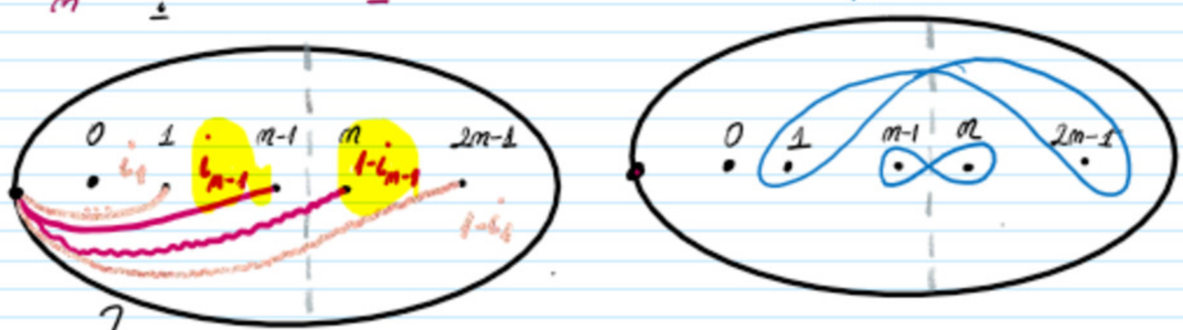
<u>Proof</u> For $i_1,\dots,i_{m-1}\in\{0,\dots,N-1\}$

$\tilde{\mathcal{U}}_{\underline{i}}$ $\mathcal{G}_m^N$

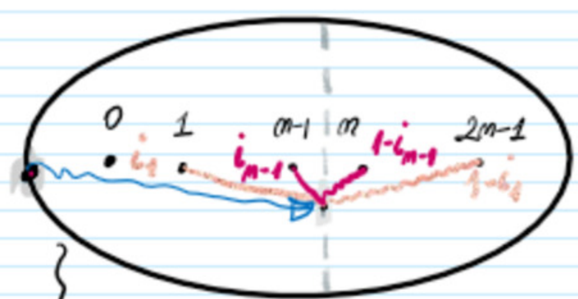


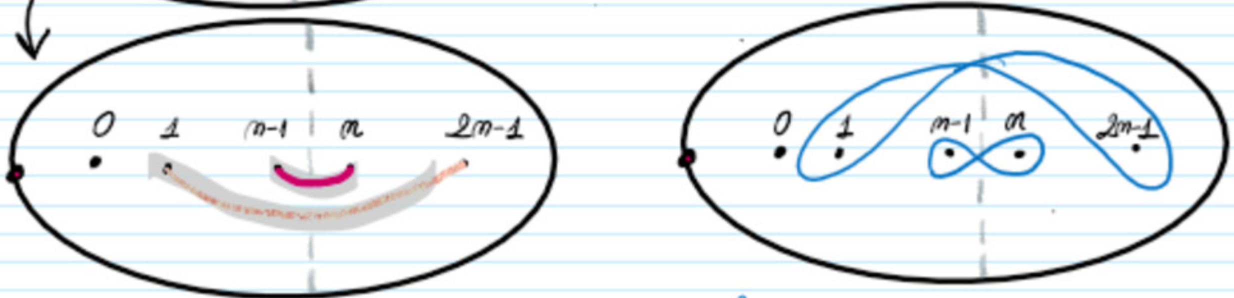
$\boxed{N=2}$ Jones polyn. $i_1,\dots,i_{m-1}\in\{0,1\}$

$\mathcal{E}_m^2=\sum_{\underline{i}} d^{-\sum i_k}\,\tilde{\mathcal{U}}_{\underline{i}}$ $\mathcal{G}_m^2$



up to homotopy and d-coefficients





$\mathcal{E}_m^2$ fork and $\mathcal{G}_m^2$ noodle

• **Corollary 2** (ADO invariants from $\mathbb{Z} \oplus \mathbb{Z}_{2N}$-covering spaces)

The N^{th} coloured Alexander invariant comes from an intersection pairing in a $\mathbb{Z} \oplus \mathbb{Z}_{2N}$-covering of a conf. sp. in the punctured disk.

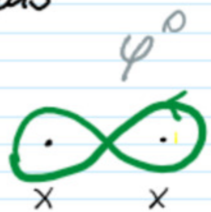
Questions

1. Model with embedded Lagrangians
2. Simple lifts (paths η to the base point)
3. Suitable for computations

III Model with embedded Lagrangians

Construction Ideas

$(k=n)$

①② Replace 

by 

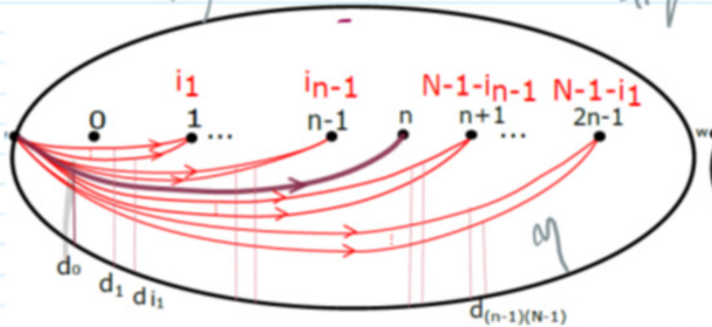
(change the local system)

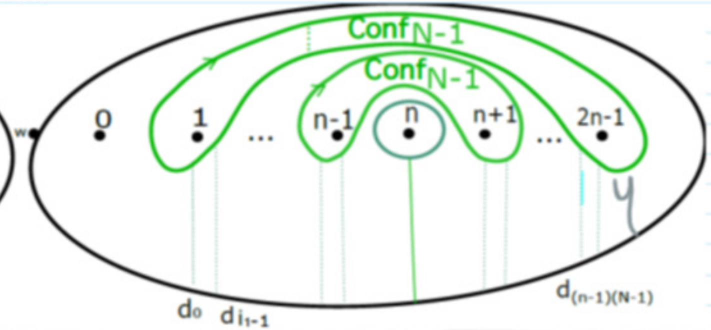
Add an extra puncture

• Def (Homology classes) $\underline{i}=(i_1,\dots,i_{n-1})$, $i_j\in\{0,\dots,N-1\}$

$\mathcal{F}_{\underline{i}}\in H^{-n}_{2n,\,(n-1)(N-1)+1}$ $\qquad$ $\mathcal{L}_{\underline{i}}\in H^{-n,\partial}_{2n,\,(n-1)(N-1)+1}$

punctures $\qquad$ # particles





• Th 2 (A'20 Unified model through embedded Lagrangians)

$$\boxed{\Lambda_N(\beta_n):=u^{-w(\beta_n)}\cdot u^{-(n-1)}\sum_{i_1\dots i_{n-1}=0}^{N-1}\left\langle(\beta_n\cup 1\!\!1_n)\,\mathcal{F}_{\underline{i}},\,\mathcal{L}_{\underline{i}}\right\rangle}\in\mathbb{Z}[x^{\pm1},d^{\pm1},u^{\pm1}]$$

Then :
$$\begin{cases}\boxed{J_N(L,q)=\Lambda_N(\beta_n)\,/\,\psi_{1,q,N-1}}\\[1ex]\boxed{\Phi_N(L,\lambda)=\Lambda_N(\beta_n)\,/\,\psi_{1-N,\xi_N,\lambda}}\end{cases}$$

specialisations of coefficients

• **Corollary**:

$$
\begin{tikzcd}
 & \text{Col. Jones polyn. } J_N(L) & \\
\begin{array}{c}\text{Immersed state sum}\\ \Omega_N(\beta_n)\end{array} \arrow[ur, "\psi_{1,2N-1}"] \arrow[dr, "\psi_{1-N,\xi_N,\lambda}"'] & & \begin{array}{c}\text{Embedded state sum}\\ \Lambda_N(\beta_n)\end{array} \arrow[ul, "\psi_{1,2N-1}"'] \arrow[dl, "\psi_{1-N,\xi_N,\lambda}"] \\
 & \text{Col. Alexander polyn. } \Phi_N(L) &
\end{tikzcd}
$$

Figure eights

Circles

Th 1

Th 2

• **Corollary** $\boxed{N=2}$: Jones and Alexander polyn. from the same geometric/topological picture

• **Example** T = trefoil knot:

$n=2$ $\quad \beta_2 = \sigma^3$ $\quad T$

$N-1=1$

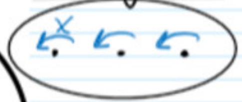
$i_1 \in \{0,1\}$ $\quad \mathbb{Z}(\sigma^3 \cup \mathbb{1})$

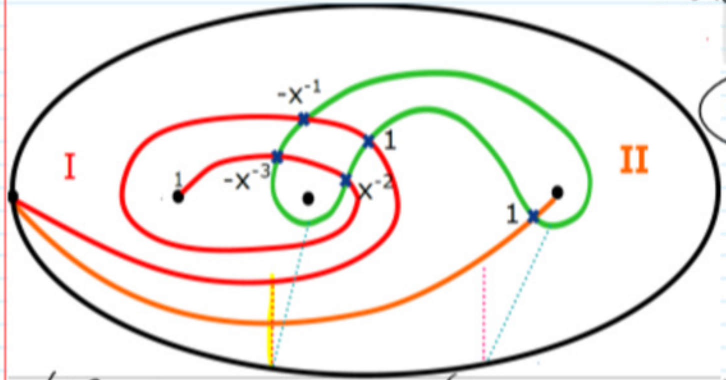
(I) $\mathcal{F}_1$ $(i_1=1)$

(II) $\mathcal{F}_0$ $(i_1=0)$

Th2 Embedded model

Monodromy of the local system





$((\sigma^3 \cup \mathbb{1})\,\mathcal{F}_1, \mathcal{L}_1)$ $\quad ((\sigma^3 \cup \mathbb{1})\,\mathcal{F}_0, \mathcal{L}_0)$

$\in \mathbb{Z}[u^{\pm 1}, x^{\pm 1}, d^{\pm 1}]$

$$\boxed{\Lambda_2(\sigma^3) = u^{-4}\left(d^{-1}(-x^{-3}+x^{-2}-x^{-1}+1)+1\right)}$$

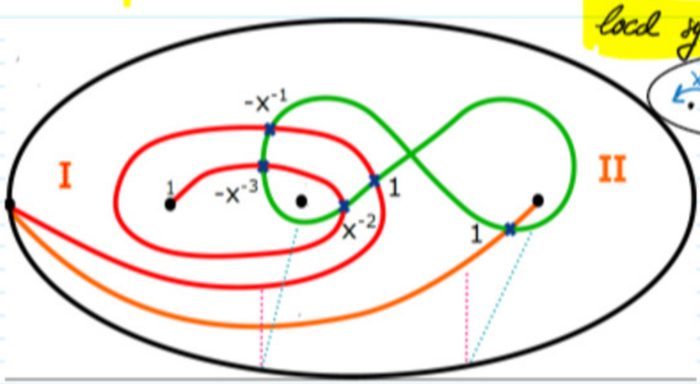
$u=q^2$, $x=q^2$; $d=q^{-2}$ $\qquad$ $u=\xi_2^{-\lambda}$, $x=\xi_2^{2\lambda}$; $d=\xi_2^{-2}=-1$

$$\boxed{J(T) = -q^{-8}+q^{-2}+q^{-6}} \qquad \boxed{\Delta(T,x) = x-1+x^{-1}}$$

Jones $\qquad$ Alexander

Th1 Immersed model

Monodromy of the local system



$$\boxed{\Omega_2(\sigma^3) = u^{-4}\left(d^{-1}(-x^{-3}+x^{-2}-x^{-1}+1)+1\right)}$$

$u=q^2$, $x=q^2$; $d=q^{-2}$ $\qquad$ $u=\xi_2^{-\lambda}$, $x=\xi_2^{2\lambda}$; $d=\xi_2^{-2}=-1$

$$\boxed{J(T) = -q^{-8}+q^{-2}+q^{-6}} \qquad \boxed{\Delta(T,x) = x-1+x^{-1}}$$

Jones $\qquad$ Alexander