Khovanov homology via Floer theory of the 4-punctured sphere

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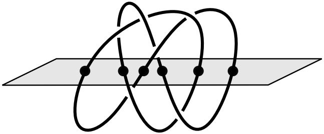
(1) $\mathrm{Kh}(K;\mathbb{Q})$ via Floer theory of Hilbert schemes [Seidel-Smith'06, Manolescu'06, Abouzaid-Smith'16'19]

Input: n-bridge decomposition of a knot



$$\begin{array}{ccc} T_1 & \longmapsto & L(T_1) = (S^2)^n \\ & & \downarrow \\ (S^2, 2n) & \longmapsto & \mathcal{M}(S^2, 2n)^{4n} \\ & & \uparrow \\ T_2 & \longmapsto & L(T_2) = (S^2)^n \end{array}$$

Output: $\mathrm{Kh}(K) \cong \mathrm{HF}(L(T_1), L(T_2))$ Lagrangian Floer homology

Applications? Hard to obtain, because the moduli space $\mathcal{M}(S^2, 2n)^{4n}$ is really complicated

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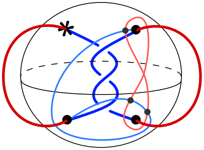
(2) $\widetilde{\mathrm{Kh}}(K;\mathbf{k})$ via immersed curves [K-Watson-Zibrowius'19]

Input: tangle decomposition $K = T_1 \cup_{(S^2,4\mathrm{pt})} T_2$, which is not required to be 2-bridge

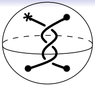
Output: $\widetilde{\mathrm{Kh}}(K) \cong \mathrm{HF}(\gamma_{\mathrm{Kh}}(T_1), \gamma_{\mathrm{BN}}(T_2))$
Floer homology of immersed curves in $(S^2, 4\mathrm{pt})$

Applications? Yes! Because the Floer theory of the surface $\mathcal{M}^{\mathrm{new}} = (S^2, 4\mathrm{pt})$ is well-understood



$\widetilde{\mathrm{Kh}}($$) = \mathbf{k}^3$

Introducing curve invariants

Input: pointed 4-ended tangle $T\colon I^* \sqcup I \hookrightarrow D^3$, e.g. 

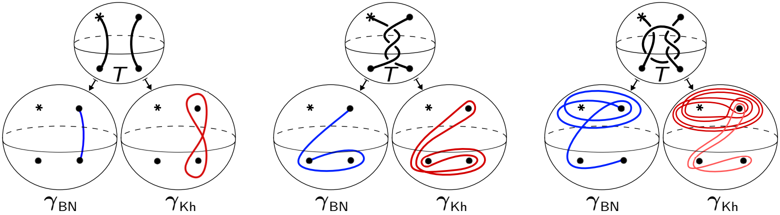
Output: two tangle invariants $\gamma_{\mathrm{BN}}(T)$ and $\gamma_{\mathrm{Kh}}(T)$,
each a collection of immersed curves in $(S^2, 4\mathrm{pt}) = \partial(D^3, T)$
considered up to regular homotopy (bigraded, with local systems)

Construction: (more on this later)

$$T \xmapsto{\text{Quantum Topology}} Д(T)^{\mathcal{B}} \xmapsto{\text{Symplectic Geometry}} \gamma_{\mathrm{BN}}(T)$$

algebraic invariant — immersed curve

Examples:



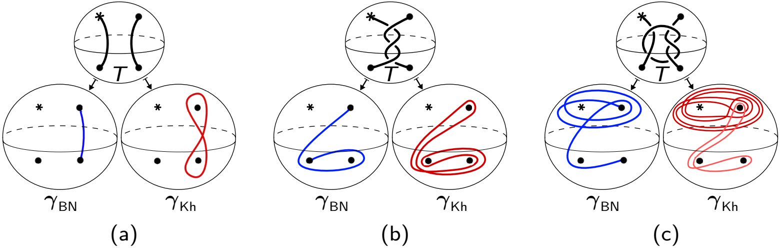
Properties of curve invariants

(1) $\gamma_{\mathrm{BN}}(T)$ is of the form $\{$one arc$\} \cup \{$compact curves$\}$

(2) $\gamma_{\mathrm{Kh}}(T)$ is of the form $\{$compact curves$\}$

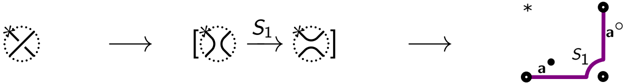
(3) Over $\mathbf{k} = \mathbb{F}_2$, naturality with respect to any braid move σ:
$\sigma(\gamma_{\mathrm{BN}}(T;\mathbb{F}_2)) = \gamma_{\mathrm{BN}}(\sigma(T);\mathbb{F}_2)$ [compare (a) and (b)]

(4) For rational tangles:
$\gamma_{\mathrm{BN}}(T;\mathbb{F}_2)=\{$embedded arc$\}$, $\gamma_{\mathrm{Kh}}(T;\mathbb{F}_2)=\{$figure eight$\}$



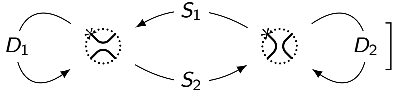
Construction of $\gamma_{BN}(T)$ (similar for $\gamma_{Kh}(T)$)

$$T \xrightarrow[\text{cube of resolutions}]{\textbf{Step 1}} Д(T)^{\mathcal{B}} \xrightarrow[\leftrightarrow \mathbf{a}^{\circ} \quad \leftrightarrow \mathbf{a}^{\bullet}]{\textbf{Step 2}} \gamma_{BN}(T)$$



Step 1. Due to [Bar-Natan'05, Khovanov'02, Manion'17 K-Watson-Zibrowius'19]

$Д(T)^{\mathcal{B}}$ is a chain complex over *deformed reduced* arc algebra:

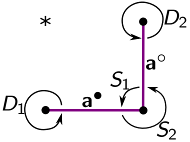
$\mathcal{B} = \mathbf{k}\Big[$  $\Big] \Big/ (D_j S_i = 0 = S_i D_j)$

Step 2. See picture above. Two points:

(a) The key miraculous coincidence we observed:

$\mathcal{B} \hookrightarrow$ *Wrapped Fukaya Category of* $(S^2, 4\text{pt})$

(b) Theorem of [Haiden-Katzarkov-Kontsevich'14 Hanselman-Rasmussen-Watson'17 K-Watson-Zibrowius'19] allows to translate $Д(T)^{\mathcal{B}} \mapsto \gamma_{BN}(T)$

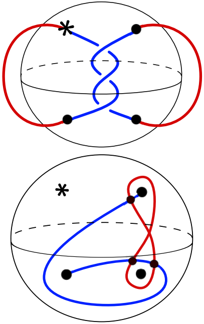


Important: Curve invariants are computable! [Thouin (torus case) Zibrowius-Chhina'19 Zibrowius'21]

Topological setup:

Conway sphere $S^2 \cap K = 4\text{pt}$, marked by $*$

$\implies$ tangle decomposition $K = T_1 \cup_{(S^2,4\text{pt})} T_2$



$\widetilde{\mathrm{Kh}}($$) = \mathbf{k}^3$

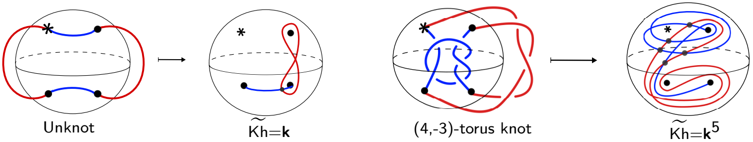
Main Theorem (K-Watson-Zibrowius'19)

Reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K;\mathbf{k})$ *is isomorphic to Lagrangian Floer homology of the curve invariants associated to the two tangles:* $\widetilde{\mathrm{Kh}}(K) \cong \mathrm{HF}\left(\gamma_{\mathrm{Kh}}(T_1), \gamma_{\mathrm{BN}}(T_2)\right)$

(*as bigraded* $\mathbf{k}$*-vector spaces*)

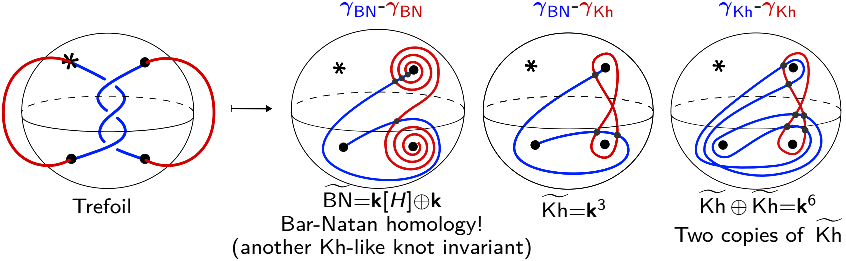
In practice: $\dim \mathrm{HF}\left(\gamma_{\mathrm{Kh}}(T_1), \gamma_{\mathrm{BN}}(T_2)\right) = \left|\gamma_{\mathrm{Kh}}(T_1) \overset{\min}{\cap} \gamma_{\mathrm{BN}}(T_2)\right|$

More examples:



Other pairings and the proof

Q. What if we take Floer homology of the same types of curves?



Proof. Wanted: $\widetilde{\mathrm{BN}}(T_1 \cup_{(S^2,4\mathrm{pt})} T_2) \cong \mathrm{HF}(\gamma_{\mathrm{BN}}(T_1), \gamma_{\mathrm{BN}}(T_2))$

$$T_1 \xrightarrow{\text{Cube of resolutions}} Д(T_1)^{\mathcal{B}} \xrightarrow{\text{Geometrization}} \gamma_{\mathrm{BN}}(T_1)$$

$$T_2 \xrightarrow{\text{Cube of resolutions}} Д(T_2)^{\mathcal{B}} \xrightarrow{\text{Geometrization}} \gamma_{\mathrm{BN}}(T_2)$$

- $\widetilde{\mathrm{BN}}(T_1 \cup_{(S^2,4\mathrm{pt})} T_2) \cong H_*[\mathrm{Mor}(Д(T_1), Д(T_2))]$
- $H_*[\mathrm{Mor}(Д(T_1), Д(T_2))] \cong \mathrm{HF}(\gamma_{\mathrm{BN}}(T_1), \gamma_{\mathrm{BN}}(T_2))$

We extended the Geometrization Theorem of Haiden-Katzarkov-Kontsevich to the morphism level

Applications

Theorem. [K-Lidman-Moore-Watson-Zibrowius; in progress]
Consider a non-trivial knot K with a strong inversion $\tau \curvearrowright (S^3, K)$, $\tau^2 = \text{id}$. If $S_r(K)$ and $S_{r'}(K)$ are $\mathbb{Z}/2$-equivariantly diffeomorphic surgeries, then $r = r'$. In other words:

$\mathbb{Z}/2$*-equivariant version of Cosmetic Surgery Conjecture* holds.

Theorem/Conjecture. [K-Watson-Zibrowius; in progress]
$\widetilde{\mathrm{Kh}}(K; \mathbb{Q})$ is mutation invariant. (Known over $\mathbb{F}_2$.)

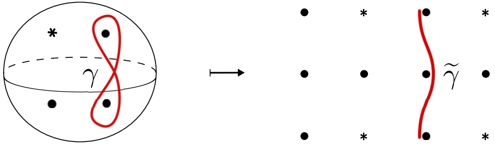
Conjecture.
Existence of an essential Conway sphere in (S^3, K) implies $\dim \widetilde{\mathrm{Kh}}(K; \mathbb{F}_2) \geqslant 17$.

The key to all of these results is *the geography question*:

Which immersed curves can be components of $\gamma_{\mathrm{Kh}}(T)$?

Geography of curve invariants

- Consider the cover $(\mathbb{R}^2 \smallsetminus \mathbb{Z}^2) \to (\mathbb{T}^2 \smallsetminus 4\text{pt}) \xrightarrow{/\mathbb{Z}_2} (S^2, 4\text{pt})$
- Will study curves $\gamma \looparrowright (S^2, 4\text{pt})$ via their lifts $\tilde{\gamma} \looparrowright \mathbb{R}^2 \smallsetminus \mathbb{Z}^2$



Geography Restriction Theorem (K-Watson-Zibrowius'21)

If γ is a component of $\gamma_{\text{Kh}}(T;\mathbb{F}_2)$, then $\tilde{\gamma}$ is a line of slope p/q (modulo twining about the punctures). Furthermore, up to the action of $MCG(S^2 \smallsetminus 4pt)$, γ is equal to either $r_n(0)$ or $s_{2n}(0)$ for some $n \geqslant 1$. The local systems are all trivial.

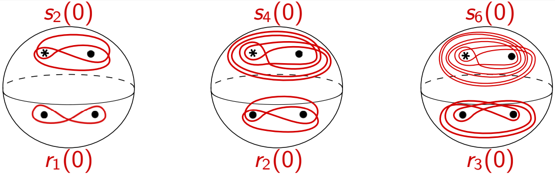
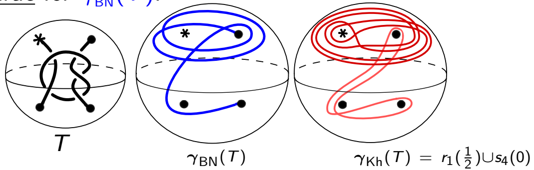
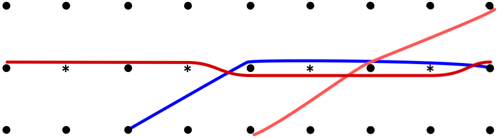


Illustration: Curves in $\gamma_{Kh}(T)$ are figure eights lifting to lines.
The same is <u>not true</u> for $\gamma_{BN}(T)$.





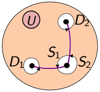
Key ideas:

(1) Linearity of a curve $\Longleftarrow$ Fishtails cancel out

(2) Deal with fishtails over .: using $\gamma_{Kh} = Cone(\gamma_{BN} \xrightarrow{H} \gamma_{BN})$

(3) Deal with fishtails over $*$ by constructing
an extension to A_∞ algebra $\mathcal{B}^*[U] = WrFuk(S^2 \smallsetminus .:;\ \text{relative } *)$

$$\underset{(a)}{M(T)^{\mathcal{A}}} \xrightarrow[\text{(b)}]{\text{HMS}} \boxed{\text{Д}(T)^{\mathcal{B}^*[U]}} \xrightarrow{U=0} \text{Д}(T)^{\mathcal{B}} \xrightarrow{\text{1-1}} \gamma_{BN}$$



$\mu_4(D_1, S_2, D_2, S_1) = U$

(a) Matrix factorizations [Khovanov-Rozansky'08]

(b) $\mathcal{A} \leftrightarrow \mathcal{B}^*[U]$ via homological mirror symmetry for $S^2 \smallsetminus$.: [Abouzaid-Auroux-Efimov-Katzarkov-Orlov'13]

Thank you!