Kh can be thought of as the combinatorial part of homologies satisfying a sheaf rela.

\[
\begin{array}{c}
\chi \\
\downarrow \\
\chi
\end{array} \quad \text{LES}
\]

First eq.

\[ E_2 = H_{\text{red}}(K) \rightarrow \hat{H}(\Sigma(K)) = E_\infty \]

\[ \text{chain groups have } \frac{1}{2} \text{ the dim of usual chain groups.} \]

\[ \Sigma(K) = \mathbb{M}^3 \cup K \]

\[ \mathbb{M}^3 \times \mathbb{Z} \]

\[ S^3 \mid K \]

\[ \Sigma(K) \]

\[ 2:1 \text{ except at } K \]

\[ S^3 \]

\[ \mathbb{Z} \rightarrow \mathbb{Z}^2 \]
These are each filling of some 3-manifold w/ torus boundary.

Fiber homologies satisfy a surgery exact sequence

\[ \widehat{HF}(\Sigma(x)) \to \widehat{HF}(\Sigma(\infty)) \]

Figure out what is \( \widehat{HF}(\Sigma(\text{unknot})) \).
and figure out

\[ \hat{HF}(\Sigma(00)) \longrightarrow \hat{HF}(\Sigma(0)) \]

one sees that these mimic the (reduced) Khovanov chain complex.

\underline{Instanton knot Floer homology (over \(\mathbb{Z}\))
(Kronheimer-Mrowka)}

\[ Kh \cong I^k \]

Chain groups of \(I^k \) are
(roughly speaking) given by the homology of a space of flat \(SU(2)\) connections on \(S^3\backslash K\).
\[ \text{Rep}(K) = \left\{ \rho : \pi_1 \left( S^3 \setminus K \right) \to \text{su}(1,2) : \text{tr}(\rho(n)) = 0 \right\} \]

To specify a \( \rho \), we give a pt on \( S^2 \) for each meridian to the arcs of a knot diagram.
\[ \text{Rep}(\mathcal{G}) \]

\[ \mathbb{S}^2 \]

\[ H_* \rightarrow \mathbb{Z} \oplus \mathbb{Z} \]

\[ \text{KH}^{**}(\mathcal{G}) \]

\[ n_1 = n_2 = n_3 \]

\[ \text{UTS}^2 \cong \text{IRIP}^3 \]

\[ \mathbb{Z} \oplus \mathbb{Z}/2 \]

\[ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \]

See work of Lewallen.
Esoteric eg (L. - Watson) over (after Couture), $F = \mathbb{R}/\mathbb{Z}$.

Knots w/ "strong involution" (rotational sym. v/ $\mathbb{Z}$ fixed pts).

So $d_T := \text{id} + T$ is a differential $d = dk + d_T$ is also a differential (exercise).
Call \((C, d)\) the complex \(\text{v/homology } H\).

\(d\) preserves \(d\), changes \(i\) by less than or equal to 1.

Get a filtration \(F^i C^j(D)\) \(\text{v/E}_2\) page \(E_2 = H^{ii, i}\)

Define a new grading \(k\)

This grading is preserved by \(d_T\), raised by \(d_k, k\). Create a filtration \(G^k C^j\) \(\text{w/ first differential in spec seq } d_T\). The \(E_3\) page and later are invariants.
Double filtered, \( i \)-graded homology.

Triple graded associated graded homology

\( Gr^i Gr^k H^j (K) \).

Triple grading can be used (e.g.) to distinguish strong inversions in the same knot.