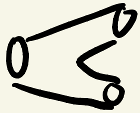
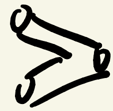
$1 \longmapsto v_+$



$v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+$

$v_- \longmapsto v_- \otimes v_-$



$v_+ \otimes v_+ \longmapsto v_+$

$v_+ \otimes v_- \longmapsto v_-$

$v_- \otimes v_+ \longmapsto v_-$

$v_- \otimes v_- \longmapsto 0$

$v_+ \longmapsto 0$

$v_- \longmapsto 1$

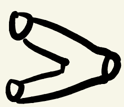
$$\mathbb{C} \xrightarrow{2} \mathbb{C}$$

1+1 dim'l TQFTS are given by Frobenius algebras.

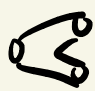
Ring w/ coproduct + counit.
(Satisfying some axioms).

$$R = \mathbb{C}[x]/x^2 = \langle 1, x \rangle_{\mathbb{C}} \qquad \begin{array}{l} 1 \longleftrightarrow v_+ \\ x \longleftrightarrow v_- \end{array}$$

$\mathbb{C} \xhookrightarrow{i} R$



$R \otimes R \xrightarrow{m} R$



$R \xrightarrow{\Delta} R \otimes R$

$R \xrightarrow{\varepsilon} \mathbb{C}$.

The axioms are exactly the rules that make this the building block of TQFT.

What if we replace $\mathbb{C}[x]/x^2$ with some other Frobenius algebra?

(would need such a Frobenius algebra to have $\dim_{\mathbb{C}} = 2$ so that χ is an invariant) [Exercise].

eg.

$$R = \mathbb{C}[x]/(x-\alpha)(x-\beta) \qquad \alpha, \beta \in \mathbb{C}$$

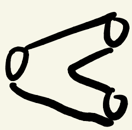
$\alpha = \beta \qquad E_2 = Kh^{i,j} = E_\infty$

Lee homology.

$\alpha \neq \beta \qquad E_2 = Kh^{i,j} \qquad E_\infty = \mathbb{C} \oplus \mathbb{C}$

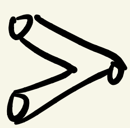
(when K is a knot).

Focus on $\mathbb{C}[x]/x^2-1$



$v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+$

$v_- \longmapsto v_- \otimes v_- + v_+ \otimes v_+$



$v_+ \otimes v_+ \longmapsto v_+$

$v_+ \otimes v_- \longmapsto v_-$

$v_- \otimes v_+ \longmapsto v_-$

$v_- \otimes v_- \longmapsto v_+$

New differential $\widetilde{d}$ on $\widetilde{CKh}$ preserves or raises the quantum j-degree.

$$F^j \widetilde{CKh}^i(D) := \bigoplus_{k \geq j} CKh^{i,k}(D)$$

$$\cdots \subseteq F^j \widetilde{CKh}^i(D) \subseteq F^{j-1} \widetilde{CKh}^i(D) \subseteq \cdots$$

$$\widetilde{d}\left(F^j \widetilde{CKh}^i(D)\right) \subseteq F^j \widetilde{CKh}^{i+1}(D)$$

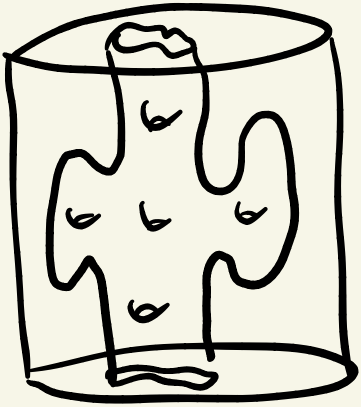
Turns out that $\bigoplus_i \widetilde{HKh}^i(D)$ has $\dim_{\mathbb{C}} = 2$ (when K is a knot).

$\exists s \in \mathbb{Z}$ s.t.

$$Gr^{s-1} \widetilde{H}^0 \cong Gr^{s+1} \widetilde{H}^0 \cong \mathbb{C}.$$

This s is called the Rasmussen invariant.

By working at the cochain level.



$$K_0 \hookrightarrow \mathbb{R}^3 \times \{0\}$$

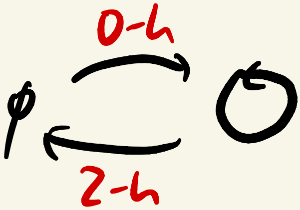
$$\Sigma \xhookrightarrow{\text{smooth}} \mathbb{R}^3 \times [0,1]$$

compact connected ortd

$$K_1 \hookrightarrow \mathbb{R}^3 \times \{1\}$$

Link cobordism.

Such a link cobordism can be presented as a "movie" of diagram stills/frames. Successive diagrams differ by R'm'r or handle move:



Exercise: Convince yourself that each handle move $D \to D'$ gives an obvious chain map $C(D) \longrightarrow C(D')$ using the TQFT.

There are also chain homotopy equivalences for each R'm'r move.

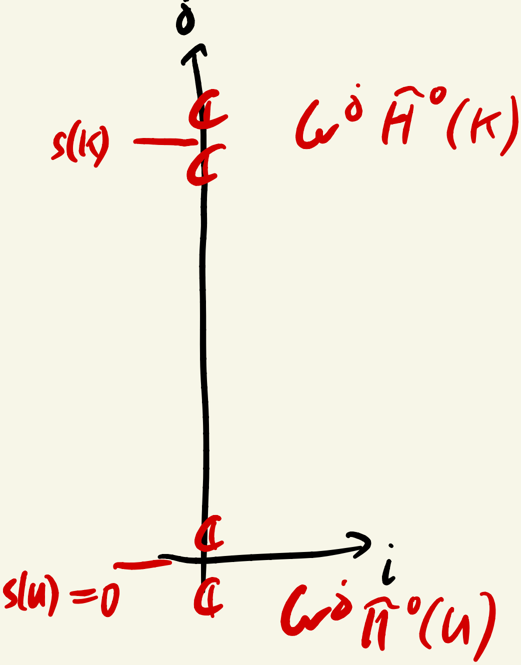
So this gives a map

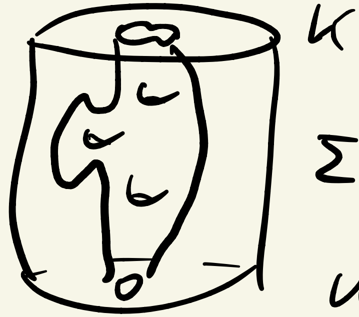
$$HKh^{i,j}(K_0) \longrightarrow HKh^{i,j+\chi(\Sigma)}(K_1)$$

$$\mathcal{F}^j \widetilde{H}^i(K_0) \longrightarrow \mathcal{F}^{j+\chi(\Sigma)} \widetilde{H}^i(K_1).$$

By arguing at the cochain level one can show

$$\begin{array}{ccc} \widetilde{H}^0(K_0) & \xrightarrow{\cong} & \widetilde{H}^0(K_1) \\ \| & & \| \\ \bigcup_j \mathcal{F}^j \widetilde{H}^0(K_0) & & \bigcup_j \mathcal{F}^j \widetilde{H}^0(K_1) \\ \| & & \| \\ \mathbb{C} \oplus \mathbb{C} & & \mathbb{C} \oplus \mathbb{C} \end{array}$$





If $\chi(\Sigma) = 0$ and $s(K) > 0$ then we would have a contradiction.

In fact we see

$$s(K) \leq |\chi(\Sigma)|$$

s gives rise to a good lower bound on the "slice genus" (minimal distance to the unknot).

Exercise ① Go to knot atlas.

Pick a knot. Look at $HKh^{i,j}$.

Compute s.

$$HKh^{i,j} = E_2$$

$$E_\infty = Gr^j \widehat{H}^i$$

② Look at Piccirillo's celebrated paper on Conway knot for a larger example.

---

sl_n also has a TQFT.

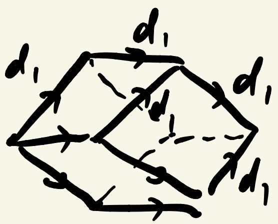
$$\bigcirc \qquad R = \mathbb{C}[x]/x^n$$

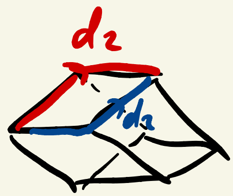
replace with other deg $= n$ monic polynomial. Get a spectral sequence starting at sl_n homology.

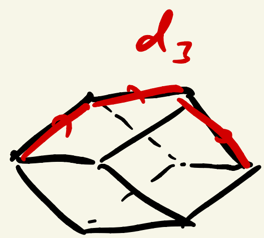
Advertize Baldwin-Hedden-L.

"**Thm**" Sensible perturbations of Khov differential give rise to knot invariants taking the form of spectral sequences.

work over $\mathbb{F} = \mathbb{Z}/2$.







$$\tilde{d} = d_1 + d_2 + d_3 + \cdots$$

d_1 = Khov diff.

Homologically filtered, j-graded.

$E_2 = Kh^{i,j}(D)$.

Q: $E_2 = E_\infty$?