What is (co)homology?

C (fin. dim.) vector space (over $\mathbb{C}$ or $\mathbb{Z}/2 = \mathbb{F}$)

or a free abelian group.

$d : C \to C$, $C \circlearrowleft d$

$d^2 = 0 \iff \operatorname{im} d \subseteq \ker d$

$H^*(C) := \ker d / \operatorname{im} d$.

Quite often $C = \bigoplus_{i \in \mathbb{Z}} C^i$

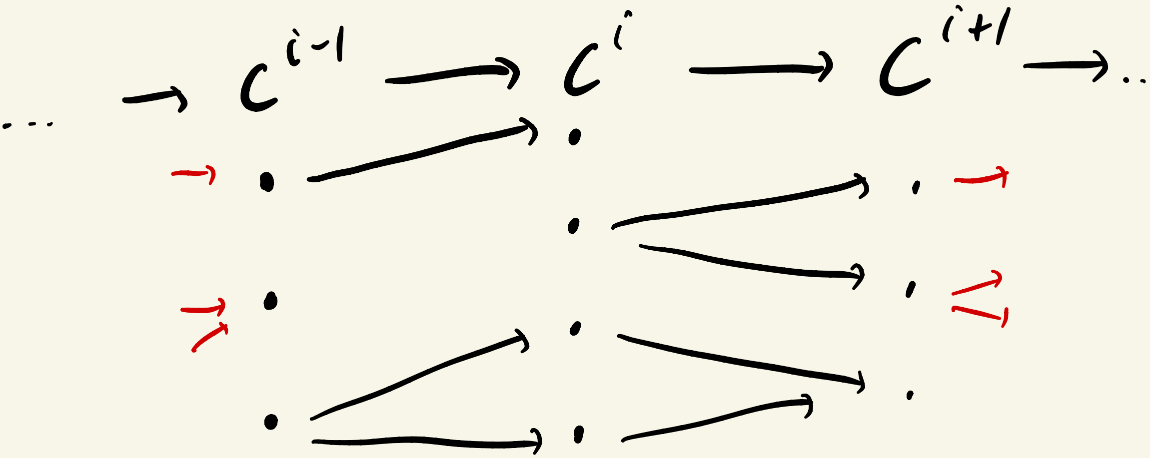
and $d(C^i) \subseteq C^{i+1}$

(cohomological grading)

$d = \bigoplus_{i \in \mathbb{Z}} d^i \quad d^i : C^i \to C^{i+1}$

$H^i(C^*) = \ker d^i / \operatorname{im} d^{i-1}$

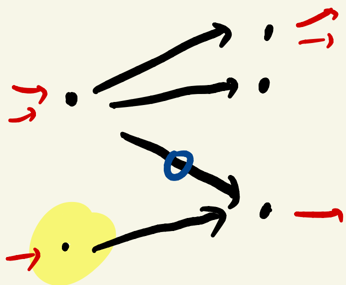
Over $\mathbb{F}$, it's easy to draw a picture of (C^*, d)



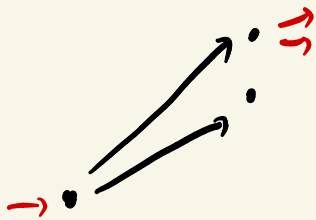
Dots: basis elements.

Two dots are joined by an arrow if the relevant matrix entry for d^i is 1.

We move to a simpler complex (fewer dots) by this rule:



Remove arrow, add arrow for each zig-zag, remove double arrows.



Gauss elimination.

The result of doing this is a smaller complex that is (exercise) chain homotopy equivalent to the original complex.

Keep eliminating until we arrive at a complex w/ no arrows.
In other words $d = 0$

$$H^*(C, d=0) \cong C$$

We've now computed cohomology.

Filtrations

(All v.s. over $\mathbb{F}$, fin. dim).

$$C = \bigcup_{j \in \mathbb{Z}} F^j C$$

fin. dim.

$$\cdots \subseteq F^{j-1}C \subseteq F^j C \subseteq F^{j+1} C \subseteq \cdots$$

Sometimes these inclusions are reversed.

$$d \circlearrowright C = \bigcup_{j \in \mathbb{Z}} F^j C$$

$$d^2 = 0$$

This induces a filtration on the cohomology of C.

$$[a] \in F^j H^*(C)$$

$$\Longleftrightarrow$$

$$\exists b \text{ w/ } b \in F^j C, \quad [a] = [b].$$

Often we want to compute the filtration of H^*C (up to isomorphism).

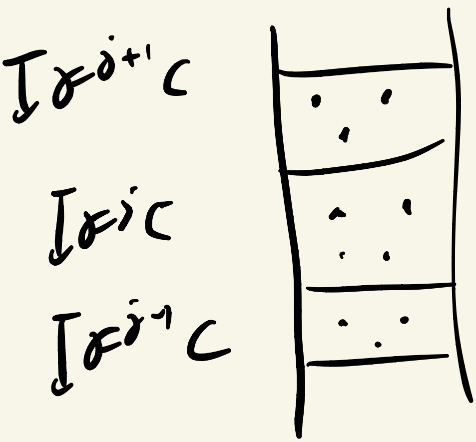
$$\ldots \subseteq F^j H^*(C) \subseteq F^{j+1} H^*(C) \subseteq \ldots$$

The jump in dimension is measured by $Gr^j H^*(C) := F^j H^*(C) \big/ F^{j+1} H^*(C)$.

$$H^* \cong \bigoplus_j Gr^j H^*$$

To draw pictures it helps to pick a "graded basis" for $C = \bigcup_{j \in \mathbb{Z}} \mathcal{F}^j C$

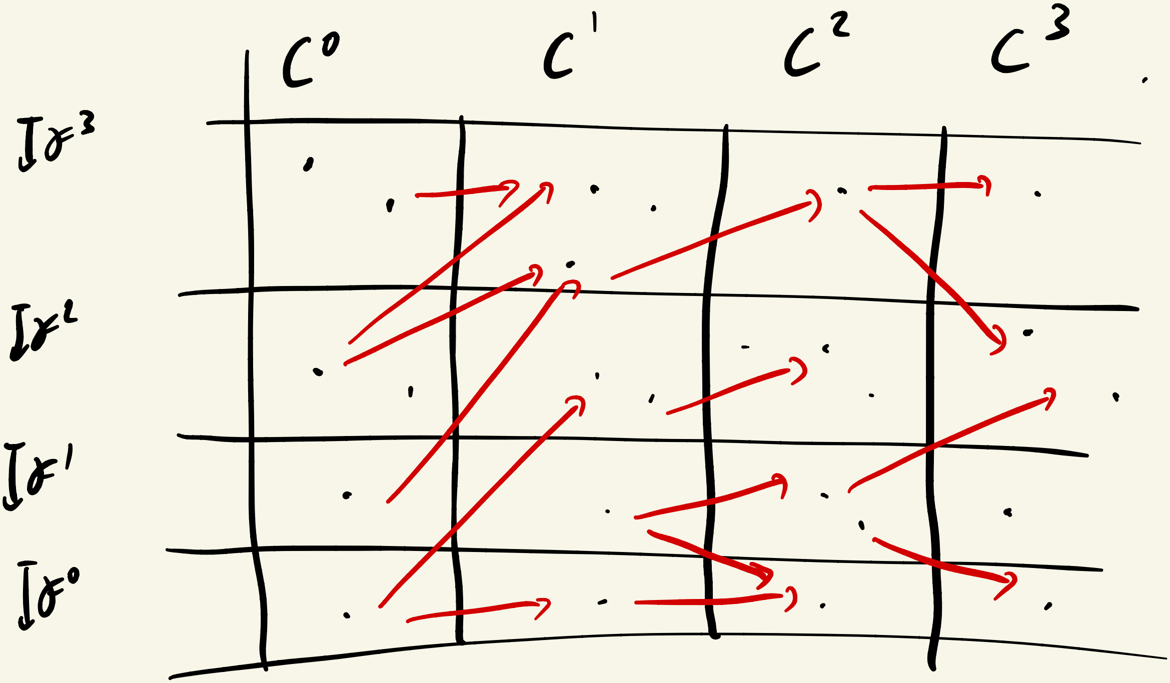
$$\mathcal{F}^j C = \mathcal{F}^{j-1} C \oplus \langle g_1^j, \dots, g_{n_j}^j \rangle$$



We assume for ease that there is a cohomological grading.

$$C = \bigoplus_{i \in \mathbb{Z}} C^i \qquad C^i = \bigcup_{j \in \mathbb{Z}} \mathcal{F}^j C^i$$

$$d(C^i) \subseteq C^{i+1}$$



Procedure: Eliminate all arrows of highest slope $\nearrow$. Then arrows of next highest slope. Continue until all arrows have gone.

You have computed $Gr^j H^i$.

The successive pictures obtained by eliminating all arrows of a certain slope are called __pages__ of a spectral sequence.

Usually $d(\mathcal{F}^j C^i) \subseteq \mathcal{F}^j C^{i+1}$

Page E_1 chain complex of horizontal arrows. $CKh^{i,j}(D)$.

$E_2 = H^*(E_1)$ chain complex of arrows of slope -1. $Kh^{i,j}(K)$

$E_3 = H^*(E_2)$ . . .

?

E_∞ terminal page.

$C^* \xrightarrow{\varphi} D^*$ $\quad \varphi(\mathcal{F}^j C^i) \subseteq \mathcal{F}^j D^i$

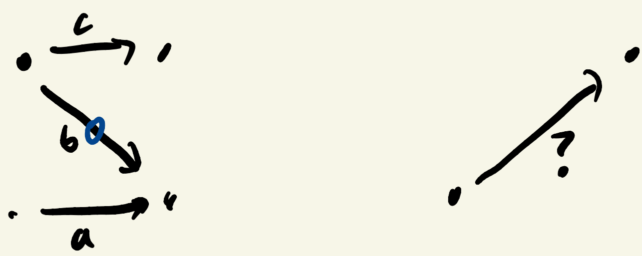
Induced map on the spectral sequences.

$$\varphi_k : E^{i,j}_{C^*,k} \xrightarrow[\text{map}]{\text{chain}} E^{i,j}_{D^*,k}.$$

k^{th} page.

How does Gauss elimination work not over $\mathbb{F}$? (over $\mathbb{C}$?).

Decorate arrows w/ field elements.



Over $\mathbb{Z}$?. $\quad \bullet \xrightarrow{2} \bullet$

$\ldots 0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \ldots$ f.g.

Grp

$0, \quad \mathbb{Z}, \quad \mathbb{Z}/2, \quad 0$