Exercises (Rasmussen Lectures 1 and 2)

(1) If you don't know about the elliptic fibration on a K3 surface, find someone who does. If you do, explain it to someone. Show that the complement of a regular fiber is simply connected. (Hint: sections). Deduce that the manifold M_K from the first lecture is simply connected and has the same cohomology ring as M. (By Freedman's theorem, this is enough to show they are homeomorphic.)

(2) If you don't know why the Bar-Natan complex of the 3 crossing tangle from the first lecture is homotopy equivalent to a complex of the form $q^5T_0 \to q^3T_0 \to qT_0 \to T_1$ (where T_0 and T_1 are the two generating objects), find someone to explain it to you. If you do, find someone to explain it to. What are the boundary maps in this complex? Note that this complex is a chain in the sense discussed at the end of today's lecture. Find someone who can explain how this complex relates to a curve in the 4-punctured sphere.

(3) Let X be the wedge of n circles. Find a 2-dimensional cell complex homotopy equivalent to $\mathrm{Sym}^2 X$. (Start with the usual cell complex structure on $X \times X$.)

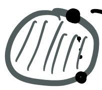
(4) Let γ be an immersed curve in T^2 (either closed or with endpoints on ρ_-). By retracting to a neighborhood of the 1-handles,, show that γ can be written as an iterated mapping cone of the generating objects L_0 and L_1 (assuming the geometric description of the mapping cone from lecture.)

(5) An immersed closed curve or arc in a surface S is unobstructed if it lifts to an embedded curve in the universal cover of S. Let $\widehat{S}$ be D^2 with 4 red arcs on its boundary. Find all isotopy classes of unobstructed arcs in S. What if $\widehat{S}$ is the annulus with one red arc on each boundary?

ex: (1+1)-dim: $S^1 \mapsto A \in \mathrm{Vect}_{\Bbbk}$ $\quad \left(\Rightarrow (S^1)^{\sqcup k} \mapsto A^{\otimes k}\right)$

 $\mapsto \begin{matrix} A \\ \uparrow \\ A\otimes A \end{matrix}$,  $\mapsto \begin{matrix} A \\ \uparrow \\ \Bbbk \end{matrix}$, ... + rels

$\rightsquigarrow$ A is a comm. Frobenius alg.

ex: dim( $\}2k$) $= \frac{1}{k+1}\binom{2k}{k} =: C_k$ "Catalan numbers"

ex: embedding $\Sigma_1 \hookrightarrow \Sigma_2 \Rightarrow \mathbb{C}(q)$-linear map $TL(\Sigma_1) \to TL(\Sigma_2)$

ex: $\mathbb{A} \hookrightarrow \mathbb{D} \Rightarrow$

$$\begin{matrix} TL(\mathbb{A}) & \longrightarrow & TL(\mathbb{D}) \\ \cong & & \cong \\ \mathbb{C}(q)[x] & \longrightarrow & \mathbb{C}(q) \end{matrix}$$

← show this + find the map

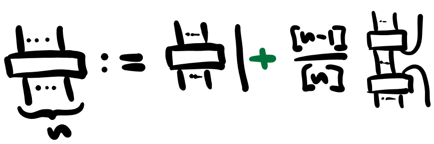
ex: check

$$V(n) = \begin{tikzcd} \underset{-n}{\bullet} \arrow[r, bend right, "{[1]}"'] & \underset{2-n}{\bullet} \arrow[l, bend right, "{[n]}"'] & \cdots & \underset{n-4}{\bullet} \arrow[r, bend right, "{[n-1]}"'] & \underset{n-2}{\bullet} \arrow[l, bend right, "{[2]}"'] \arrow[r, bend right, "{[n]}"'] & \underset{n}{\bullet} \arrow[l, bend right, "{[1]}"'] \end{tikzcd}$$

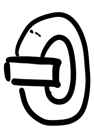
"Weyl module" is a $U_q(sl_2)$-module

$\left(K(\underset{i}{\bullet}) = q^i \underset{i}{\bullet}\right)$

ex: explizitly decompose $V(1)\otimes V(2) \cong V(3) \oplus V(1)$

ex: verify 

is an idempotent

ex: show  $= (-1)^n [n+1]$

ex: (1+1)-dim: $\quad S^1 \mapsto A \in \mathrm{Vect}_{\mathbb{k}} \quad \left(\Rightarrow (S^1)^{\sqcup k} \mapsto A^{\otimes k}\right)$

 $\mapsto \begin{matrix} A \\ \uparrow \\ A \otimes A \end{matrix}$,  $\mapsto \begin{matrix} A \\ \uparrow \\ \mathbb{k} \end{matrix}$, ... + rels

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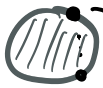
ex: $\dim\Big($  $\Big\}2k\Big) = \frac{1}{k+1}\binom{2k}{k} =: C_k$ "Catalan numbers"

ex: embedding $\Sigma_1 \hookrightarrow \Sigma_2 \Rightarrow \mathbb{C}(q)$-linear map $TL(\Sigma_1) \to TL(\Sigma_2)$

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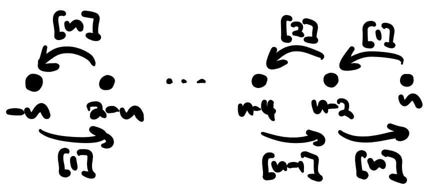
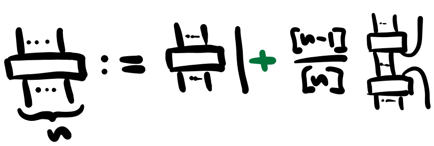
$\leftarrow$ show this + find the map

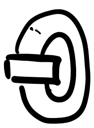
ex: check $V(n) =$ 

"Weyl module" is a $U_q(\mathfrak{sl}_2)$-module

$\left(K(i) = q^i\, i\right)$

ex: explicitly decompose $V(1) \otimes V(2) \cong V(3) \oplus V(1)$

ex: verify  is an idempotent

ex: show  $= (-1)^n [n+1]$

ex: @ $r=3$, compute $V(1)\otimes V(2) \not\cong V(3)\oplus V(1)$ but **is** filtered by $V(1)$, $V(3)$

$\left(V(3) \hookrightarrow V(1)\otimes V(2) \twoheadrightarrow V(1)\right)$

ex: @ $r=3$, $V(3) \not\cong V(3)^\vee$ action via antipode

$S(e) = -ek^{-1}$
$S(f) = -kf$
$S(k^{\pm}) = k^{\mp}$

ex:  $\in \mathcal{N}$

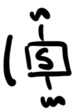
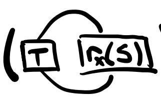
ex: $\mathcal{N}$ is a 2-sided monoidal ideal

ex: $BN(\Sigma, p)$ is graded via $\deg(\alpha) = \frac{1}{2}|p| - \chi(\alpha)$

ex: $\bigcirc \cong q\,\phi \oplus q^{-1}\phi$ in $BN(\mathbb{D}, \phi)$

ex: $BN(\mathbb{D},\phi)^{\oplus} \cong \mathrm{Vect}_{\mathbb{k}}^{\mathbb{Z}}$

ex: describe $BN(S^2)$

ex: $\mathrm{Hom}_{BN_m^n}\Big($  ,  $\Big) \cong q^{\frac{m+n}{2}}\, Kh\Big($  $\Big)$

ex: find d for  $:= \cdots \xrightarrow{d} q^5$  $\xrightarrow{d} q^3$  $\xrightarrow{d} q$ 

and show "categorical idempotence"  $\simeq$ 

ex: same for  $:= \cdots \to q^5$  $\to q^3$  $\to q$  $\to ||$

ex: $\mathrm{Hom}\Big($  ,  $\Big) \neq 0$